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A RESPONSE OF THE ECONOMY TO CHANGES IN EMPLOYMENT STRUCTURE

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Abstract

This paper examines the response of economies to shocks in demand. The analysis is performed by using a general equilibrium model with monopolistic competition in hi-tech sectors, perfect competition in a traditional sector, labor market frictions, and bargaining that determines wages. We posit that labor market frictions result in unemployment in the equilibrium. Under a positive shock in demand, new individuals become unemployed. If the elasticity of substitution between varieties of differentiated goods is a decreasing/increasing function, then the inter-sector wage inequality enlarges/falls. In the first case, all individuals gain but those who lost their jobs.

Non-technical summary

This paper examines the response of economies and, in particular, inter-sector wage inequality to shocks in demand. The analysis is performed by using a general equilibrium model with monopolistic competition in hi-tech sectors and perfect competition in a traditional sector. Hi-tech sectors are horizontally differentiated with respect to labor skills required by production technologies and to efficiency of these technologies. Motivated by larger wages, workers intend to work in hi-tech sectors but face a risk to be unemployed. Unemployment appears as a consequence of job market frictions: rejected skilled workers fail to find a new job immediately. The wages of employed workers are agreed through a bargaining mechanism.

As a result of monopolistic competition, the sector size is affected not only explicitly by consumers' distribution of spending but also implicitly by the elasticity of demand in the sector. Redistribution of spending from traditional to hi-tech goods underlies the expansion of hi-tech sectors. This creates additional supply and demand for workers. We posit that the supply exceeds the demand: a part of job market candidates is rejected, and the number of unemployment agents increases. *Variability of the elasticity of substitution (VES) between hi-tech goods implies* that a demand shift in favor of a *single* differentiated product affects the production of *all* sectors. The influence is based on *changes in the relative diversity* of the differentiated products, which is the number of the product's varieties normalized by spending of consumers for this product. We argue that the economy can respond in two alternative ways: the relative diversity of the differentiated products either enlarges or shrinks. In the first case, the demand for specific varieties of more diversified goods goes down. Having a limited monopoly power, firms higher price their goods compensating the shift in demand. Whole sectors incur a negative scale effect: the redistribution of the output to larger amount of firms with a drop of individual productions reduces the sector output. Sector i, which produces the favored differentiated product, overcomes this effect and increases its output because of a direct influence of increase in spending for its goods. Following the output, the labor increases (decreases) in sector i (respectively, in the other sectors) but the number of employees in each firm decreases. Hiring less workers, firms value each of them more and agree to increase their wages through the bargaining mechanism. Eventually, the inter-sector wage differential enlarges. We explored numerically that all individuals who keep their job status become better off; only the welfare of new unemployed agents drops. In the second case, when the relative diversity shrinks, we end up with the reverse prediction regarding the inter-sector wage differential, claiming that it decays.

The two responses of an economy are distinguished with consumers' elasticity of substitution σ between hi-tech goods. A decreasing σ leads to an increasing relative diversity of the differentiated product, and vice versa. Intuition underlying this result is rather simple. We argue that spending additional money for specific goods, consumers exhibit a more elastic demand. The latter follows a growth/decline in demand for specific goods if σ is increasing/decreasing.

In our model, changes in the wage inequality are explored as a size effect, which is washed out under preferences with constant elasticity of substitution. On the other hand, the existence of an equilibrium is proved under preferences with relatively small VES. Nevertheless, the response of the wage inequality to a shift in tastes can be distinguishable. To our knowledge, this is the first attempt to estimate quantitatively the effect of VES in structural models with monopolistic competition. The scale of the response positively correlates with the efficiency of technologies in the corresponding industries, whose model proxy is the ratio of fixed to variable costs faced by firms.

Contents

1 Introduction

A remarkable rise of the income dispersion in the US and UK since the 1960s and 1970s respectively and more complex changes of the income inequality in other developed and developing countries have been discussed in many papers (paper [Machin, 2008] is a recent review). Explaining evolution of income inequalities economists attribute to the "race" between the relative demand for labor and the relative supply of it, skill-biased technological changes, possibly, international trade, and the role of labor market institutions (see, respectively [Katz and Murphy, 1992, Acemoglu, 2002, Wood, 1995, Lee, 1999] among others). Up to 40% of the income redistribution in favor of high-skilled workers occurs as a result of inter-industry changes [Berman et al., 1998]. Explanations of these changes are less conclusive, despite basic arguments remain the same. In addition, researchers indicate the role of the integral development of industries [Gittleman and Wolff, 1993] and the structure of consumers' demand [Leonardi, 2003].

We are going to contribute to this analysis theoretically, further exploring the role of consumers' demand under exogenous variations in preferences of individuals. The objective of this paper is to develop a general equilibrium model (GEM) with monopolistically competing firms to describe the response of an economy to an exogenous shift in consumers' tastes for skill-intensive products. This response will include changes in the inter-sector income inequality, the number of unemployed agents, re-distribution of sector sizes, and welfare of individuals. Since size effects are absent in economies characterized by preferences with a constant elasticity of substitution (CES), we construct a model economy with a variable elasticity of substitution (VES), following Zhelobodko et al. [2012]. The approach of Zhelobodko et al. [2012] is extended to a multi-sector economy, assuming frictions in the labor market and bargaining that results in identical expected wages in different sectors. This economy consists of several hi-tech sectors, in which differentiated goods are produced, and of homogeneous numéraire sector. The labor market is defined, assuming frictions (described in details, f.i., in a review by Rogerson et al. [2005]) in hi-tech sectors that prevent rejected high-skilled job candidates to find another job immediately. Frictions together with a special searching procedures allow Helpman et al. [2008] to explain matching between high-skilled workers and efficient firms earlier studied by Amiti and Pissarides [2005].

We show that under a shift in consumer tastes for skill-intensive products, an expansion of corresponding hi-tech sectors *a priori* can occur in two alternative ways. The first way is described by a more competitive monopolistic competition: each firm decreases its prices and increases the output. Then the expansion of sectors occurs at firm as well as at sector levels. This requires an excessive inflow of workers but their wages follow the prices for sectors' goods (since labor is under pressure of unemployed agents). As a result, the income inequality goes down. The second opportunity strengthens the monopolistic nature of the monopolistic competition. The equilibrium variables mentioned above move in the opposite directions. Our paper suggests that an economy "chooses" its response to a shift in consumer tastes depending on consumers' elasticity of substitution between hi-tech goods in such a manner that an increasing elasticity σ leads to a more competitive monopolistic competition, and vice versa. The intuition underlying this result is rather simple. Spending additional money for specific goods, consumers exhibit a more elastic demand. The latter follows a growth/decline in demand for specific goods if σ is increasing/decreasing. According to Zhelobodko el al (2012), economies characterized by an increasing/decreasing σ, weaken/strengthen the competition in hi-tech sectors under exogenous shocks that relax trade barriers. We argue that those economies exhibit the opposite changes in the competition when the expansion of hi-tech sectors is driven by the preferences of consumers.

The case of an increasing σ is supposed to be more adequate [Ottaviano et al., 2002, Zhelobodko et al., 2012]. We introduce a family of the corresponding utilities and estimate numerically the response of individuals' welfare. Under changes of preferences in favor of hi-tech goods, all individuals gain but those who lost their jobs.

Following Zhelobodko et al (2012), we consider preferences with relatively small variability. Nevertheless, the response of the wage inequality to a shift in tastes can be large. The scale of the response depends on the efficiency C of technologies in the corresponding industries, whose model proxy is the ratio of fixed to variable costs faced by firms. Changing in preferences involving goods that are produced by industries with higher values of C affects the inter-industry wage inequality stronger.

The novelty of our tackling of the wage inequality problem is in the introduction of industries with monopolistically competing firms. Namely such a structure of supply highlights the role of the elasticities of demand explored in this paper.

Our research has links to various papers. Some efforts aim to describe trends in the development of hi-tech sectors and relate one to another. As debated in many sources, technological progress shifts the labor demand to favor skilled workers in OECD countries (f. e., Machin and Reenen [1998]). Less skilled manufacturing workers move to other sectors, mainly to service sectors [Berman et al., 1998]. Alongside this trend, relative prices for manufacturing output have been dropping in several developed countries with Australia, US, and Canada as prominent examples [Baldwin and MacDonald, 1998, Ricaurte, 2010, Lowe, 2011]. The causality of these trends remains unclear. For example, Loupias and Sevestre [2013] conclude that variations of labor costs affect producer price changes in France. By no means trying to list all possible origins of price changes, we mention trade liberalization [Baldwin and MacDonald, 1998], exchange rates, business cycles [Messina et al., 2009], inflation [Gagnon, 2009], and macro shocks [Clark, 2006].

The evolution of hi-tech sectors and changes in inequality of workers' income are investigated by using dynamical models. Based on classical models of endogenous growth [Judd, 1985, Grossman and Helpman, 1991, 1994], Acemoglu [2012] links a development of hi-tech sectors to appearance of new innovative products. Initially, their production requires labor of skill workers. Within time innovations are adopted by industries. Eventually, standardized firms competing for production of those adopted products win over more hi-tech firms because they face lower labor costs employing less skilled workers. Hi-tech firms, in turn, switch to other innovations. To estimate changes in gap between wages of skilled and unskilled workers Acemoglu [1997] models the competition between them and finds that this gaps can both increase and decrease depending on frictions on labor markets. In particular, the inequality shrinks if searching of new jobs is impeded by labor market frictions and the process of matching is subdued.

Research methods relate our paper to general equilibrium modeling with monopolistically competing firms. Original Dixit–Stiglitz setting of monopolistic competition involves consumers' preferences with a constant elasticity of substitution [Dixit and Stiglitz, 1977]. As a result, an equilibrium markup does not depend on the size of an economy. Improving this unrealistic model outcome, Ottaviano et al. [2002] construct a model with a quasi-linear utility and incorporate a pro-competitive response of the equilibrium to an enlargement of the economy: outputs increase and prices decrease. Behrens and Murata [2012] endow consumers with a sub-utility of the "constant absolute risk aversion" (CARA) type and explore other economies with this pro-competitive response. They decompose the gain from trade into gains from product diversity and gains from pro-competitive effects. Zhelobodko et al. [2012] describe a class of utilities that lead to this pro-competitive response in terms of the elasticity of demand between hi-tech goods. An influence of the elasticity of demand on effective tax policies and an income dispersion is derived theoretically by Di Comite et al. [2013] and Osharin et al. [2014]. Melitz [2015] link the Zhelobodko et al. [2012] framework to his trade theory involving heterogeneous firms [Melitz, 2003, Melitz and Redding, 2012] and propose an empirical strategy that separates effects coming from the supply and demand sides to firms' profit. Empirical evidence that variable elasticity of demand affects profit would overcome the criticism of VES general equilibrium modeling given by Bertoletti and Epifani [2014].

We built our analysis on recent several theoretical contributions that aims to explain links between market size, income inequality, unemployment, and consumer tastes. Behrens and Robert-Nicoud [2014] proposed a model, in which market size disproportionally favors more skilled agents. They conclude that an enlargement of markets can increase the wage differential through selection effect: the selection process is slower on larger markets and this weakens the competition between workers. Based on a UK data, Leonardi [2003] derives empirically that more educated individuals are hired for high-skilled jobs and consume more skill-intensive goods than less educated individuals. Departing from this finding, he claims that an increase in supply of high-skilled workers (coming from new college graduates) also moves up the demand for them through the shift of the aggregate consumers' demand. This idea is supported by a simple general equilibrium model. We argue that the structure of consumers' demand matters by its own. Namely, changes in the demand structure affects both the supply of and the demand for labor, and the integral effect is ambiguous.

2 Model

2.1 Supply Side

We consider an economy that consists of a homogeneous sector (henceforth sector 0) with perfect competition and of n high-tech sectors with N_i monopolistically competing single-product firms in each sector, $i = 1, \ldots, n$. In the homogeneous sector firms price their products at the marginal cost because of the perfect competition. Assigning for simplicity 1 to productivity in this sector, we have that prices p_0 are equal to wages w_0. We denote L_0, L_1, …, L_n the number of employed workers in sectors 0, 1, …, n. Let L_{n+1} be the number of unemployed workers.

A firm producing a good ξ_i in the i-th hi-tech sector faces sector specific fixed c_i^φ costs. Its variable costs are associated with wages of its employees, which work with an inverse productivity c_i^v, homogeneous within sector i. This firm tunes its price $p(\xi_i)$ to maximizes the profit:

$$\pi(\xi_i) = p(\xi_i)Q(\xi_i) - c_i^v Q(\xi_i)w(\xi_i) - c_i^\varphi w(\xi_i) \longrightarrow \max, \tag{1}$$

where $Q(\xi_i)$ is the aggregate demand for the good ξ_i in the i-th sector that depends on the price $p(\xi_i)$.

Accepting standard monopolistic competition settings, we expect each particular firm to be so small that tuning its prices it fails to affect a price index in the i-th sector. The number (mass) N_i of goods and firms in the i-th sector is regulated by the free entry condition

$$\pi(\xi_i) = 0.$$

2.2 Demand Side

Upper-tier choice. The aggregate demand is based on individual choices of consumers. The income of consumers depends on their jobs; namely on the sector where they work. As a result, there are $n+2$ types of the incomes y_0, y_1, …, y_n, y_{n+1} in the economy, denoted after the sectors; index $n+1$ describes unemployed agents. A consumer with an income y_j decides upon her demand in two steps. First, she differentiates between hi-tech goods represented by consumption indices H_i, $i = 1, \ldots, n$, and a homogeneous good H_0. Maximizing the upper-tier Cobb–Douglas utility

$$U = H_0^{\beta_0} H_1^{\beta_1} \ldots H_n^{\beta_n} \longrightarrow \max, \tag{2}$$

where the exponents β_i, $i = 0, 1, \ldots, n$, are summed up to 1, the consumer allocates her income proportionally to the exponents β_i for products of the i-th sectors.

Lower-tier choice. Second, each consumer "reminds" the definition of the consumption index as an additive utility from the consumption of each particular product $\xi_i \in [0, N_i]$, $i = 1, \ldots, n$. Consumers choose the demands $q_j(\xi_i)$, where $j = 0, \ldots, n+1$ indicates the income y_j of the consumer, for each $i = 1, \ldots, n$ maximizing the consumption index

$$H_i = \int\limits_{N_i} u_i\left(q_j(\xi_i)\right) d\xi_i \longrightarrow \max, \tag{3}$$

with a low-tier utility function $u_i(\varkappa)$, which reflects preferences for sector i's differentiated product, subject to budget constraint

$$\int\limits_{N_i} p(\xi_i) q_j(\xi_i) d\xi_i \leq \beta_i y_j. \tag{4}$$

Elasticity of substitution between hi-tech goods. The following function

$$\sigma_i(\varkappa) = -\frac{u_i'(\varkappa)}{u_i''(\varkappa)\varkappa},$$

interpreted as the elasticity of the substitution between hi-tech goods, underlies the optimal individual demands $q_j(\xi_i)$. Namely, varying the first order conditions of optimization problem (3), (4) with respect to the price $p(\xi_i)$ for this particular good, the firm finds that the elasticity $\mathtt{E}_{p(\xi_i)} q_j(\xi_i)$ of the demand $q_j(\xi_i)$ with respect to its price $p(\xi_i)$ is opposite to the elasticity of substitution:

$$\mathtt{E}_{p(\xi_i)} q_j(\xi_i) = -\sigma_i(q_j(\xi_i)). \tag{5}$$

Exploring a multi-sector economy, we introduce an aggregate elasticity of substitution

$$\mathfrak{S}(\xi_i) = \sum_{j=0}^{n+1} \frac{q_j(\xi_i) L_j}{Q(\xi_i)} \sigma_i(q_j(\xi_i)), \tag{6}$$

where $Q(\xi_i)$ is the aggregated demand $Q(\xi_i) = \sum_{j=0}^{n} q_j(\xi_i) L_j$. By definition, this aggregate elasticity of substitution is a weighted average of the individual elasticities of substitution. One can easily check that property (5) is extended to $\mathfrak{S}(\xi_i)$:

$$\mathtt{E}_{p(\xi_i)} Q(\xi_i) = -\mathfrak{S}(\xi_i), \tag{7}$$

Therefore, the first order conditions of consumers' optimization problem state that the price elasticity of the aggregate demand $Q(\xi_i)$ is described with the aggregate elasticity of substitution $\mathfrak{S}(\xi_i)$. We will use this $\mathfrak{S}(\xi_i)$ to explore the equilibrium of the model.

Examples. The simplest example of preferences is given by a power function u. These preferences have a constant elasticity of substitution. We introduce two families of VES preferences: one of them with $\sigma' < 0$ and the other with $\sigma' > 0$. As we will discuss later in Section 2.5, in a neighborhood of individual equilibrium demands (which is appeared to be a positive neighborhood of zero), they both are close to the sum of power functions. Nevertheless, a general form is more complicated. Aiming at a simple expression for the elasticity of substitution σ, we put

$$u(\varkappa) = \ln\left(\varkappa + 1 + \sqrt{\varkappa^2 + 2\varkappa}\right). \tag{8}$$

This leads to $\sigma(\varkappa) = 2 + 2/(\varkappa + 1)$. A more general form of the elasticity of substitution

$$\sigma(\varkappa) = A\left(1 + \frac{1}{\varkappa + 1}\right)$$

is obtained with hyperelliptic functions:

$$u(\varkappa) = \frac{A}{2(A-1)}(\varkappa(\varkappa+2))^{\frac{A-1}{A}} F_1\left(1, 2 - \frac{2}{A}, 2 - \frac{1}{A}, -\frac{\varkappa}{2}\right), \quad A > 2, \tag{9}$$

where $F_1(a, b, c, z)$ is a standard notation for hyperelliptic functions (see, f.i., Whittaker and Watson [1990] for description of hyperelliptic functions). The representation of these hyperelliptic functions by an integral is given in the Appendix. The family

$$u(\varkappa) = \frac{A}{A-1}\varkappa^{1-\frac{1}{A}}(2\varkappa+1)^{1+\frac{1}{2A}} F_1\left(1, 2 - \frac{1}{2A}, 2 - \frac{1}{A}, -2\varkappa\right), \quad A > 1, \tag{10}$$

represents preferences with an increasing elasticity of substitution:

$$\sigma(\varkappa) = A\left(2 - \frac{1}{\varkappa + 1}\right).$$

2.3 Labor Market

The production side is characterized by technologies that vary from sector to sector but not from firm to firm. Firms face sector-specific productivities $1/c_i^v$ of labor. In order to produce an optimal number $Q(\xi_i)$ of goods firms hire

$$l(\xi_i) = c_i^v Q(\xi_i) + c_i^\varphi \tag{11}$$

workers.

Firms and workers agree upon the wages through bargaining discussed in details by Stole and Zwiebel [1996]. The equal division of surplus between a firm and each of its workers underlies bargaining. The firm employs workers one-by-one estimating the gain from hiring an additional employee. Workers compare the wages proposed by the firm with the outer alternative, for a moment assumed to coincide with the wage w_0 in the standard sector. According to [Stole and Zwiebel, 1996], the wages are

$$w(\xi_i) = \left(\frac{p(\xi_i)}{c_i^v} + w_0\right)\frac{l(\xi_i)^2 - (c_i^\varphi)^2}{2l(\xi_i)^2} \tag{12}$$

(see Lemma 5), where c_i^v is the inverse productivity of workers, and the production function is $Q = (l - c^\varphi)/c^v$ (if the number of the workers exceeds the threshold c^φ; otherwise the production is zero). Equation (12) indicates that the wages increase with a growth of firms' revenue pQ and the outer wages w_0.

Workers are motivated to enter hi-tech job markets by larger wages. To be employed in each sector including homogeneous, individuals have to acquire sector specific skills. One can think that workers choose a desired sector, get required skills, and then enter the job market of the chosen sector. Homogeneous sector is able to adopt an arbitrary number of workers. On the contrary, hi-tech sector firms hire the required number of employees and reject the other applications. We assume that the labor market exhibits some frictions. Because of them, rejected candidates cannot find a job (in a short run) in another sector (including homogeneous) because they are not qualified for it. We will show later (Lemma 9 in the Appendix) that hi-tech sector firms will not hire any unemployed agent (associated with their sector and therefore qualified for the job) for an arbitrary positive wage because then their profit decreases.

Homogeneity of technologies inside sectors equalizes intra-sector wages $w_i = w(\xi_i)$. Otherwise, firms have incentives to hire underpaid workers from other firms or unemployed agents (they have enough qualification and agree to get a smaller compensation).

We denote $\mathcal{L}$ the number of workers in the economy. It consists of employed L_i and unemployed workers L_i^u in each hi-tech, $i = 1, \ldots, n$, sector and of L_0 workers in the homogeneous sector. Symbol L_{n+1} denotes the total number of unemployed workers: $L_{n+1} = L_1^u + \ldots + L_n^u$.

A flat tax with some rate $\alpha \in (0, 1)$ is applied to the wages of all employed workers (including workers employed in the homogeneous sector) and distributed equally between unemployed agents as an unemployment benefit. This sets the (netto) income of employed in the high-tech sectors, employed in the homogeneous sector and unemployed workers to

$$y_i = (1-\alpha)w_i,\ i = 1\ldots,n, \quad y_0 = (1-\alpha)w_0, \quad y_{n+1} = \frac{\alpha(L_0 w_0 + L_1 w_1 + \ldots + L_n w_n)}{L_{n+1}} \tag{13}$$

respectively.

The choice of the labor market is assumed to be balanced on average. Given the probabilities $L_i/(L_i + L_i^u)$ and $L_i^u/(L_i + L_i^u)$ to be employed and unemployed respectively in sector i, $i = 1, \ldots, n$, agents face identical *expected* incomes that coincide with the incomes got by workers employed in the homogeneous sector:

$$\frac{y_i L_i}{L_i + L_i^u} + \frac{y_{n+1} L_i^u}{L_i + L_i^u} = y_0. \tag{14}$$

Equation (14) agrees with the choice w_0 of the alternative wages that are used in Equation (12).

The diversity of sectors, market frictions, and the balance of average incomes stay behind unemployment as a part of general equilibrium. Workers, aiming at larger wages, apply for more compensated jobs but face a risk to remain unemployed. The balance is attained on average.

2.4 Description of Model Variables. Assumptions

The primitives of the economy are sector specific utilities $\{u_i(\cdot)\}_{i=1}^n$, variable $\{c_i^v\}_{i=1}^n$ and fixed $\{c_i^\varphi\}_{i=1}^n$ costs, the total number $\mathcal{L}$ of workers in the economy, and the exponents β_0, β_1, ..., β_n of the Cobb–Douglas upper tier utility. Given those primitives, we describe a general equilibrium in the model.

The set of prices $\{p(\xi_i)\}$, individual demands $\{q_j(\xi_i)\}$, the number N_i of firms, incomes y_i, and the number L_j of workers in each sector $(i = 1, \ldots, n,\ j = 0, 1, \ldots, n,\ \xi_i \in [0, N_i])$ constitute a *general equilibrium*, if they satisfy FOC of firms' optimization problem (1), FOC of consumers' optimization problem (2),(4), budget constraint (4), the free entry condition, and labor market features (11)–(13). Simplifying notation, we keep old names $p(\xi_i)$, $q_j(\xi_i)$, and so on for the equilibrium variables.

At the beginning, the required assumptions are introduced.

Assumption 1. We will consider only monotonous differentiable functions $\sigma_i(\varkappa)$ and assume that

$$\sigma_i(\varkappa) > 1, \quad i = 1, \ldots, n, \tag{15}$$

for any $\varkappa \geq 0$.

Assumption 2. Let L_j be the equilibrium number of employed workers in sector j. We assume that this number $L_j \geq 1$, and

$$|\sigma_i'(\varkappa)| < \frac{L_j}{2C_i} \quad \text{for all } i = 1, \ldots, n,\ j = 0, \ldots, n+1, \text{ and } \varkappa \geq 0, \tag{16}$$

where $C_i = c_i^\varphi / c_i^v$ is the ratio of the fixed to variable costs.

Assumption 3. We also assume that the diversity of the equilibrium individual demands is limited:

$$\frac{\sigma_i(q_j)}{\sigma_i(q_{j'})} < 2 \quad \text{for any } j \text{ and } j'. \tag{17}$$

If σ_i increases, we assume additionally that there exists some $\delta_i > 0$ such that inequalities

$$\sigma_i'(q_{ij})q_{ij} < \delta_i < \sigma_i(q_{ij}) - 1, \quad i = 1, \ldots, n, \tag{18}$$

is valid for all equilibrium individual demands q_{ij}, $i = 1, \ldots, n$, $j = 0, \ldots, n+1$.

Zhelobodko et al. [2012] used an analogue of Assumptions 1 and 2 to prove the existence and uniqueness of the equilibrium in a single-sector economy. The number of workers in their economy (which corresponds to the quantity L_j in Equation (16)) is a model primitive. In the multi-sector economy explored here, the set of L_j, $j = 1 \ldots, n$, is endogenous, and Assumption 2 is given in a conditional but straightforward form. It links together three quantities: the equilibrium number of employed workers in each sector, variability of the elasticity of substitution, and the ratio $C_i = c_i^\varphi / c_i^v$ of fixed to variable costs. Small values of C_i are consequences of low fixed or/and high variable costs. Both effects can be attributed to less efficient economies. Therefore inequality (16) is valid if either sectors are sufficiently large (in terms of L_j), or preferences weakly deviate from the CES form, or the economy is not developed enough. We are able to present a stronger version of Inequality (16), which involves only model primitives:

$$|\sigma_i'(\varkappa)| \le \min\left\{\frac{(1-\alpha)\beta_j}{\mu_j}\left(1-\frac{1}{\delta_j}\right), (1-\beta_0)\left(1-\frac{1}{\max_k \mu_k}\right)\right\}\frac{\mathcal{L}}{2C_i},$$

assuming additionally that each σ_i is separated from 1 by some value δ_i: $\sigma_i(\varkappa) - 1 > \delta_i$, $i = 1, \dots, n$, $\varkappa > 0$.

Assumption 3 is most restrictive. It appears because the economy is multi-sector. Inequality (17) is also written with equilibrium variables. A relevant sufficient condition written in terms of model primitives is, in general, significantly more restrictive and therefore not shown here. We stress that Inequality (17) is valid if σ exhibits a small variability, which is in line with Assumption 2, or the equilibrium individual demands are weakly scattered. In our opinion, Assumption 3 is vital for the construction of the equilibrium with unspecified utility u, but any counterexample is presented. Nevertheless, we will see that for economies with large enough number $\mathcal{L}$ of agents Assumption 3 follows from Assumption 2.

Utilities (9) and (10) satisfy Assumptions 1 and 3 by construction. Assumption 2 is valid, if the number of workers in each sector is large.

2.5 Equilibrium and its Properties

Proposition 1. *Let Assumptions 1–3 be satisfied. Then a general equilibrium exists, and it is unique. This equilibrium is symmetrical with respect to varieties in the i-th differentiated product: $p(\xi_i)$ and $q_j(\xi_i)$ depends on i but not on specific varieties. We denote*

$$p_i = p(\xi_i), \quad q_{ij} = q_j(\xi_i), \quad Q_i = Q(\xi_i), \quad \mathfrak{S}_i = \mathfrak{S}(\xi_i) \tag{19}$$

the symmetrical equilibrium variables. Then they are given by the following expressions:

$$Q_i = C_i\left(\mathfrak{S}_i - 1\right), \quad i = 1, \dots, n, \tag{20}$$

$$p_i = \frac{\mathfrak{S}_i c_i^v w_i}{\mathfrak{S}_i - 1}, \quad i = 1, \dots, n, \tag{21}$$

$$q_{ij} = \frac{(1-\alpha)\beta_j Q_i}{L_j} = \frac{(\mathfrak{S}_j + 1)Q_i}{\mathcal{L}\mathfrak{S}_j}, \; j = 1, \dots, n, \tag{22}$$

$$q_{i0} = \frac{Q_i}{\mathcal{L}}, \tag{23}$$

$$q_{i,n+1} = \frac{\alpha Q_i}{L_{n+1}}. \tag{24}$$

The following Equations determine the equilibrium wages, number of employed and unemployed workers, and number of firms:

$$w_i = \frac{\mathfrak{S}_i + 1}{\mathfrak{S}_i} w_0, \quad i = 1, \ldots, n \tag{25}$$

$$L_i = (1-\alpha)\beta_i \mathcal{L} \frac{\mathfrak{S}_i}{\mathfrak{S}_i + 1}, \quad i = 1, \ldots, n, \quad L_0 = (1-\alpha)\beta_0 \mathcal{L}, \tag{26}$$

$$L_{n+1} = \mathcal{L}\left(\alpha + (1-\alpha)\sum_{j=1}^{n} \frac{\beta_j}{\mathfrak{S}_j + 1}\right), \tag{27}$$

$$L_i^u = \frac{\beta_i/(\mathfrak{S}_i + 1)}{\sum_{j=1}^{n} \beta_j/(\mathfrak{S}_j + 1)} L_{n+1}, \quad i = 1, \ldots, n, \tag{28}$$

$$N_i = \frac{(1-\alpha)\beta_i \mathcal{L}}{c_i^{\varphi}(\mathfrak{S}_i + 1)}, \quad i = 1, \ldots, n. \tag{29}$$

Wages w_0 in the homogeneous sector cannot be found in the model. One could consider the homogeneous product as a *numeraire* and assign 1 to w_0 to simplify the notation.

The idea of the Proof is standard. Under Assumptions 1 and 2, which involve only properties of the function σ_i, we derive Equation (20) from the first order condition and prove the existence and uniqueness of its solution (Lemma 3). It limits the set of admissible equilibrium prices $p(\xi_i)$ to a single value. This value of the price determines the optimal aggregate demand Q_i and elasticity of substitution $\mathfrak{S}_i$ and other variables with Equations (26)–(29). Then using the other Assumptions, we check that those variables indeed constitute the equilibrium (see details of the proof of Proposition 1 in the Appendix).

If the elasticity of substitution σ_i is a constant, then the aggregate elasticity is also a constant, and Equation (20) explicitly defines the equilibrium optimal demand. We consider a more general case of variable elasticities. Then Equation (20) defines the equilibrium optimal demand implicitly. The aggregate elasticity of substitution $\mathfrak{S}_i$, by Proposition 1, a function of i and not of ξ_i, depends on the aggregate Q_i and individual q_{ij} demands (Equation (6)). All the demands are affected by prices p_i. Therefore, firms face Equation (20) with their price as a single unknown, as expected.

The economic interpretation of Proposition 1 is in line with previous studies on the subject (see, f. i., [Helpman and Krugman, 1985]). We discuss here several issues. Equation (22), in particular, links individual demands to the size of the economy measured by $\mathcal{L}$. Under CES preferences, individuals consume less amount of specific goods, when the economy suddenly enlarges, enjoying an increasing variety of products. This amount of each good scales as $1/\mathcal{L}$. Under VES preferences, the size $\mathcal{L}$ of the economy also affects the outputs Q_i and the average elasticities of substitution $\mathfrak{S}_i$, but the influence is weak. A larger economy has a larger diversity of products, and this effect dominates the difference in outputs (if $\mathcal{L}$ is big). As a result, individuals still consume only small amount of each particular product (a rigorous proof of this statement follows from Equations (22)–(24) and (20), where the series expansion at 0 is substituted for $\mathfrak{S}$ given by Equation (6)). We claim that properties of the equilibrium are affected by the behaviour of the preferences only in a neighborhood of zero. Therefore, a restrictive part of Assumption 3, Inequality (17), is not restrictive anymore and stays in line with Assumption 2 claiming that the variability of σ is small with respect to each L_j. Indeed, the ratio (17) scales as

$$\frac{\sigma_i(0)+|\sigma_i'(0)|K_i/\mathcal{L}}{\sigma_i(0)}=1+\frac{|\sigma_i'(0)|K_i}{\sigma_i(0)\mathcal{L}},$$

where $K_i>0$ is an appropriate constant, and Assumption 3 requires that $|\sigma_i'(0)|$ is small in terms of the size $\mathcal{L}$ of the economy. Therefore ratio (17) is indeed less than 2. The primary role of diversity mentioned above is in line with the conclusion of Vives [2001]: in the first best outcome of a multi-sector economy with CES preferences, individuals face a wider diversity of goods than in a competitive equilibrium.

When $\mathcal{L}$ is large, and the equilibrium individual demands q_{ij} are small, one is interested in the specification of the preferences only in a neighborhood of zero. But in a neighborhood of zero, the hyperelliptic functions defined by (9) and (10) behave as

$$u(\varkappa)=\frac{2^{-1/A}A}{A-1}\left(\varkappa^{1-\frac{1}{A}}-\frac{A-1}{2A(2A-1)}\varkappa^{2-\frac{1}{A}}\right)+O(\varkappa^{3-\frac{1}{A}}),\quad A\geq 1,\quad \varkappa\ll 1,$$

and, respectively,

$$u(\varkappa)=\frac{A}{A-1}\left(\varkappa^{1-\frac{1}{A}}+\frac{A-1}{A(2A-1)}\varkappa^{2-\frac{1}{A}}\right)-O(\varkappa^{3-\frac{1}{A}}),\quad A>2,\quad \varkappa\ll 1,$$

do, where minus is used ahead of big-O just to stress that the next term of the series is negative. Computations are moved into the Appendix.

Equilibrium variables depend on the aggregate elasticity of substitution $\mathfrak{S}_i$. When $\mathfrak{S}_i$ is large, the demand is basically determined by the most preferred representative of the differentiated product. Then the diversity of the product is subdued and the output of specific goods enlarges as stated in Equations (29) and (20); the i-th hi-tech sector becomes similar to the homogeneous one, and the wage differential between them disappears, Equation (25). The number L_i of employed workers follows the sector size measured by $\beta_i\mathcal{L}$. The linear relationship (26) between L_i and β_i is a consequence of the Cobb–Douglas setting (2).

On the contrary, small values of $\mathfrak{S}_i$ enlarge the diversity of the differentiated product. Therefore, the quantity $1/\mathfrak{S}_i$ is interpreted as the love for variety in various papers (see Equation (29), where N_i is proportional to $1/(\mathfrak{S}_i+1)$). An expanding diversity of the differentiated product affects the total number of employed workers in a sector in the two opposite directions: the number of employees in a particular firm decreases but the number of firms increases. According to (26), the first effect is stronger. The wage bargaining balances the pressure from unemployed agents and the competition of firms for skilled workers. The second force dominates, and wages follow the size of diversity.

The number of unemployment workers expectedly increases with the taxation rate α: a growth of the unemployment benefit subdues the risk of unemployment. Tastes affect unemployment primary with development of sectors reflected by $\beta_i\mathcal{L}$ and secondary with the elasticity of demand, Equation (27). Development of sectors requires additional labor force. However, not only the demand for labor but also the supply of it changes, and the outcome of the "competition" depends on the capacity of hi-tech sectors.

The size of each sector can be measured in terms of the number of firms operated in the sector or the number of workers employed by those firms. As in other GEM, these two quantities are proportional Equations (26) and (29). The distribution of markets' size (L or N) follows the distribution of consumers' tastes β between different differentiated products.

3 Comparative Statics with Respect to Tastes

3.1 Rigorous results

Now that the general equilibrium is completely defined, we assume that tastes of consumers slowly vary. Let consumers be ready to (slightly) reduce the consumption of the homogeneous product to buy more of the i-th differentiated product: β_i increases, β_0 decreases by the same amount, and the other β_j, $j \neq 0$, $j \neq i$, do not vary.

Describing the response of the economy to those changes, we introduce a new Assumption, which is a modified version of Assumption 2.

Assumption 2$'$. Let

$$B = \max_{\varkappa, \varkappa' \in [\min\limits_{0 \leq i \leq n+1} q_{ki}, \max\limits_{0 \leq i \leq n+1} q_{ki}]} \left| \frac{\sigma_k'(\varkappa)}{\sigma_{k'}'(\varkappa')} \right|. \tag{30}$$

We assume that

$$\max_{\varkappa \in [\min\limits_{0 \leq i \leq n+1} q_{ki}, \max\limits_{0 \leq i \leq n} q_{ki}]} |\sigma_k'(\varkappa)| < \frac{L_k}{4C_k(2Bn+1)} \quad k = 1, \ldots, n. \tag{31}$$

This assumption includes the development of sectors (a small ratio of the fixed to variable costs) and a moderate deviation of the utilities from CES functions already discussed after Assumption 2. Assumption 2$'$ also requires relatively small diversity of the equilibrium demands. This requirement agrees with Assumption 3.

We denote $\mathfrak{S}^*$ the maximal equilibrium average elasticity of substitution:

$$\mathfrak{S}^* = \max_{k=1,\ldots,n} \mathfrak{S}_k. \tag{32}$$

Proposition 2. *Let Assumptions 1–3 and 2$'$ be satisfied. We also assume that the functions $\sigma_j(\varkappa)$, $j = 1, \ldots, n$, decrease or increase simultaneously and*

$$\alpha < \frac{1 - \beta_0}{2B\mathfrak{S}^* - \mathfrak{S}^* - \beta_0}.$$

Then qualitative response to an increase of β_i, $i = 1, \ldots, n$ is given by Table 1. The variables move into the opposite directions, if β_i decreases.

Table 1: Comparative statics with respect to β_i shows the response of sector k's number of firms N_k, the number of employed workers L_k, output Q_k, prices p_k, relative diversity (RD) $N_k/(\beta_k \mathcal{L})$, relative wages w_k/w_0; $k = 1\ldots, n$.

	$\beta_i \nearrow$					
	$\sigma' > 0$		$\sigma' < 0$		$\sigma' = 0$	
Response	competitive		monopolistic		neutral	
	expanding sector i	another sector k	expanding sector i	another sector k	expanding sector i	another sector k
# {firms}	$N_i \nearrow$	$N_k \searrow$	$N_i \nearrow$	$N_k \nearrow$	$N_i \nearrow$	$\frac{\partial N_k}{\partial \beta_i} = 0$
RD	$\frac{N_i}{\beta_i \mathcal{L}} \searrow$	$\frac{N_k}{\beta_k \mathcal{L}} \searrow$	$\frac{N_i}{\beta_i \mathcal{L}} \nearrow$	$\frac{N_k}{\beta_k \mathcal{L}} \nearrow$	$\frac{\partial N_i/(\beta_i \mathcal{L})}{\partial \beta_i} = 0$	$\frac{\partial N_k/(\beta_k \mathcal{L})}{\partial \beta_i} = 0$
Outputs	$Q_i \nearrow$	$Q_k \nearrow$	$Q_i \searrow$	$Q_k \searrow$	$\frac{\partial Q_i}{\partial \beta_i} = 0$	$\frac{\partial Q_k}{\partial \beta_i} = 0$
Sector output	$N_i Q_i \nearrow$	$N_k Q_k \nearrow$	$N_i Q_i \nearrow$	$N_k Q_k \searrow$	$N_i Q_i \nearrow$	$\frac{\partial (N_k Q_k)}{\partial \beta_i} = 0$
#{employed}	$L_i \nearrow$	$L_k \nearrow$	$L_i \nearrow$	$L_k \searrow$	$L_i \nearrow$	$\frac{\partial L_k}{\partial \beta_i} = 0$
Prices	$p_i \searrow$	$p_k \searrow$	$p_i \nearrow$	$p_k \nearrow$	$\frac{\partial p_i}{\partial \beta_i} = 0$	$\frac{\partial p_k}{\partial \beta_i} = 0$
Relative wages	$w_i/w_0 \searrow$	$w_k/w_0 \searrow$	$w_i/w_0 \nearrow$	$w_k/w_0 \nearrow$	$\frac{\partial (w_i/w_0)}{\partial \beta_i} = 0$	$\frac{\partial (w_k/w_0)}{\partial \beta_i} = 0$
#{unemployed}	$L_{n+1} \nearrow$		$L_{n+1} \nearrow$		$L_{n+1} \nearrow$	

Comment. If σ_i is an increasing function, then our prediction regarding the response of the number of unemployed agents is valid under the following additional assumption: the equilibrium aggregate elasticity of substitution is uniformly bounded from above:

$$\mathfrak{S}_k < \frac{16}{1-\beta_0} - 1, \quad k = 1, \ldots, n.$$

Proof of proposition 2 can be found in Lemmata 10-16.

We link an increase of spending for differentiated product i to a higher significance of product specifications for the consumer side. As soon as the elasticity of substitution $\sigma_i(q_{ij})$ between varieties of the ith good is variable, such attention of consumers to products' details makes representatives of the differentiated product closer substitutes. As a result, the elasticity $\sigma_i(q_{ij})$ increases. Then two possibilities naturally arise. If σ_i is an increasing function then the individual demands q_{ij} go up, and vise versa. The aggregate demand Q_i for the ith differentiated product will follow the individual demands.

Each average elasticity of substitution $\mathfrak{S}_k$ is obtained as a weighted average

$$\mathfrak{S}_k = (1-\alpha) \sum_{j=0}^{n} \beta_k \sigma_k(q_{kj}) + \alpha \sigma_k(q_{k,n+1})$$

of sector specific values $\sigma_k(q_{kj})$, see Equations (6) and (23). Under the demand shock ($\beta_i \longrightarrow \beta_i + \Delta\beta$, $\beta_0 \longrightarrow \beta_0 - \Delta\beta$) in favor of sector i, main changes of $\mathfrak{S}_k$ are given by the term $\Delta\beta\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big)$. Its sign is determined by the sign of the derivative σ_k'.

The integral effect of demand shocks is decomposed into two unequal parts called here primary and secondary. The primary effect is based on the redistribution of spending in favor of the ith differentiated product. This redistribution of spending drivers an expansion of sector i, increasing the number of firms/the diversity of the varieties N_i and the number L_i of employees in this sector. This effect, based on the love for variety, is observed in the economy with CES preferences, $n = 1$, and without unemployment.

Changes in the elasticity of demand measured with the aggregate elasticities of substitution $\mathfrak{S}_k$, $k = 1, \ldots, n$, underlie secondary effects, which are observed in responses of the other equilibrium variables to demand shocks. We note that the love for variety determines the *relative diveristy* $N_k/(\beta_k \mathcal{L})$ of differentiated products, which exhibits the diversity of the kth differentiated product normalized by the sector size (see, Equation (29), where this quantity is proportional to $1/(\mathfrak{S}_k + 1)$). *The secondary effects* are described with *changes* of the relative diversity of differentiated products. Under increasing/decreasing elasticity of substitution, the relative diversity of differentiated products decreases/increases. We stress that secondary effects are illuminated merely with models that involve VES preferences.

A growth of unemployment as the response to a demand shock for a differentiated product is also explained by the love for variety (i. e., relative diversity as such). A sector expansion attracts new workers. However, only a part of job market candidates is accepted. If $n = 1$ and preferences have a CES form, the fraction of unemployment agents is proportional to the sector size $\beta_i \mathcal{L}$. In a multi-industry case with VES preferences, the effect can be either amplified or subdued, depending on the relative diversity of varieties produced in the other sectors. Finally, if the expanding sector is too close to the homogeneous one ($\mathfrak{S}_i$ is large enough) and $\sigma' > 0$, then the additional demand for labor from the other hi-tech sectors prevails over the rejection rate of the expanding sector. This case is exceptional because of similarity of the expanding and homogeneous sectors.

3.2 Competitive and monopolistic responses

Zhelobodko et al. [2012] claimed that hi-tech sectors response pro-competitively to inflow of workers, increasing output and decreasing prices, only if the elasticity of substitution $\sigma(\cdot)$ is a decreasing function. A redistribution of tastes launches opposite processes, Table 1. There are no contradictions: the two models differ not only formally but also conceptually: Zhelobodko et al. [2012] described an effect of an exogenous increase of the whole economy whereas we investigate an intra-economy size effect. The general equilibrium in this paper is "more general", since the number of workers in each sector is endogenous. Its changes are induced by consumers' tastes.

Let β (slightly) increase. According to Table 1, the sector output $Q_i N_i$ and the number of firms N_i follow the tastes and also increase, as expected. This is the primary effect. At the firm level, an increasing demand for the varieties "competes" with an increasing diversity of the differentiated product. The "race" outcome is determined by the relative diversity of varieties. If the relative diversity is negative, then the diversity "wins the race", and firms' outcome Q_i increases. As stated in section 3.1, it occurs, if the preferences have an increasing elasticity of substitution (in the equilibrium).

It is worth repeating that industries react pro-competitively on changes in β. Only at the firm level, the response is ambiguous.

3.3 Response of the whole economy

We posit that a demand shock in preferences for a differentiated product i affects the production of the other differentiated products. An explicit (primary) influence of spending disappears (β_i is absent in the equations describing the equilibrium variables in sector j; see Proposition 1). Nevertheless, an implicit (secondary) effect linked to the relative diversity of differentiated products is still expected.

New opportunities in the expanding sector i attract entrepreneurs from other fields. However changes in the demand for other differentiated products are ambiguous; they are opposite to differences of the relative diversity of differentiated products. If $\sigma' = 0$, and the relative diversity of differentiated products is unchanged, then the ith sector expands at expense of the homogeneous one; workers precisely follow tastes. Let now $\sigma'_k < 0$, $k = 1 \dots, n$. Then the relative diversity of all differentiated products increases. In particular, it enlarges the diversity of the kth product, $k \neq i$ but the demand for specific varieties of the differentiated product goes down. Having a limited monopoly power, firms higher price their goods compensating the shift in demand. The whole sector incurs a negative scale effect: the redistribution of the output to larger amount of firms with a fall of individual outputs reduces the sector output. The demand for labor follows the sector output, and the number L_k, $k \neq i$, of employees decreases. The same is true at the firm level. Hiring less workers, firms value each of them more (since the linear production function has a fixed input cost) and agree to increase their wages through the bargaining mechanism. Applying the same arguments, we end up with the opposite conclusion regarding the direction of changes in the equilibrium outputs Q_k, prices p_k, the number of employees L_k, and the ratio w_k/w_0 of the relative wages, $k \neq i$, under a decrease of the relative diversity of differentiated products (occurred if $\sigma' > 0$).

As stated above, the size of secondary effects in the k-th hi-tech sector depends on changes in the relative diversity of the k-th differentiated product. Then ranking sectors with respect to these changes $\sim \partial(1/\mathfrak{S}_k)/\partial\beta_i$, $k = 1 \dots, n$ and $k \neq i$, we exhibit the elasticity of sectors' response to demand shocks. The following Lemma quantifies changes in the wage inequality.

Lemma 1. *The relative change of wages in hi-tech and homogeneous sectors is defined by the* *following formulae*

$$\left(\frac{w_j}{w_0}-1\right)^{-1}\frac{\partial}{\partial\beta_i}\left(\frac{w_j}{w_0}-1\right)=-\frac{K_{ji}\sigma_j'(\varkappa)C_j(\mathfrak{S}_j-1)}{\mathcal{L}\mathfrak{S}_i\mathfrak{S}_j} \tag{33}$$

where $j=1,\ldots,n$, $i\in\{1,\ldots,n\}$, $K_{ji}\in(1/3,2)$, $\varkappa\in(q_{j0},q_{ji})$.

Proof of Lemma 1 is given in the Appendix, Lemma 17.

New we uncover quantitative results from a technical Equation (33). Proposition 2 and Lemma 1 are valid under relatively small variability of the elasticities of substitution σ_k, $k=1,\ldots,n$, described in Assumptions 2 and 2′. However the size of the effect is proportional to this relative variability of σ_j, see the right hand side in Equation (33). We note that, given the equilibrium, Lemma 1 requires Assumption 2′ but not Assumption 2. Substituting for σ_j' its maximal feasible value given by Assumption 2′ (the second argument of the maximum in Definition (62) is assumed to be larger than the first argument), we derived that

$$\left(\frac{w_j}{w_0}-1\right)^{-1}\frac{\partial}{\partial\beta_i}\left(\frac{w_j}{w_0}-1\right)<\frac{2(1+\mathfrak{S}_j)}{\mathfrak{S}_j\mathfrak{S}_i}\frac{K_{ji}}{(2Bn+1)}.$$

Substitution $\mathfrak{S}_j=\mathfrak{S}_i=3$ serving only as an illustration fixes the first multiplier to 8/9. The estimate of the second multiplier is unclear because Assumption 2′ is rough. For example, if $n=1$, then the second multiplier is equal to 1/2 (the Proof of this statement follows from the Proof of Lemma 1), and a percentage change in the wage differential follows up to 0.44 part of percentage changes in tastes.

According to Equation (33), the wage differential is sensitive to the value of C_j, which is the ratio of fixed to variable costs. Large values of C_j are associated with developed industries (economies) so that less developed industries are more likely to suppress perturbations initiated by demand shocks. The presence of L_j in the denominator of the right hand side of (33) indicates that sectors hiring too many employees also become less sensitive to demand shocks.

3.4 Welfare

Individual welfare is understood as the indirect utility function. The latter is obtained by substitution of equilibrium variables into utility (2). There are $n+2$ types of individuals: workers of hi-tech sectors (n types), workers of the homogeneous sector, and unemployed agents. They differ by their incomes. Each type is represented by its own welfare denoted by W_0, W_1, …, W_n, W_{n+1}.

We are going to estimate qualitatively the response of welfare to a growth of spending for products of sector 1 ($i=1$ in Table 1). The expressions for the individual demands appear to be too complicated to be tackled explicitly. Therefore we use the specific family of utilities given by Equation (9) with $\sigma'<0$ (in particular, by the simplest utility (8)) and analyze changes in the welfare numerically. For the sake of simplicity we consider only two hi-tech sectors; in other words, $n = 2$. Changes in the re-distribution of spending affect the welfare explicitly, Equation (2). Namely, individuals decide to consume more products from sector 1 at expense of the homogeneous product. This decision influences the welfare positively. The welfare of hi-tech workers and unemployed agents is also affected implicitly through changes in incomes and equilibrium individual demands. The welfare pretends to follow the direction of changes in the income. According to our computer simulation, as a rule, the explicit effect dominates.

Let the welfare change from W_i to $\tilde{W}_i$, $i = 0, \ldots, n+1$, as a result of a growth of the Cobb–Douglas exponent from β_1 to $\tilde{\beta}_1$. The welfare of homogeneous sector workers are affected only by the explicit effect since w_0 does not changes. Therefore, $\tilde{W}_0 > W_0$. We proved (Proposition 2 and Table 1 for $\sigma' < 0$) that the income of workers employed in all hi-tech sectors increases but the growth of the wages in sector 1 is the largest. As a composition of two positive effects, the welfare of hi-tech workers goes up and

$$\frac{\tilde{W}_1}{W_1} > \frac{\tilde{W}_2}{W_2} > 1.$$

We find that, as a rule,

$$\frac{\tilde{W}_0}{W_0} > \frac{\tilde{W}_1}{W_1}$$

despite the wages in the homogeneous sector remain the same. The following quantitative arguments underlie the last inequality. The welfare is represented by the product of the consumption indices H_i to the power of the corresponding β_i, $i = 0, \ldots, n$, Equation (2). When β_1 increases and β_0 decreases, the ratios of the new to old welfares are primary affected by shifts in $H_1^{\beta_1}$ and $H_0^{\beta_0}$: they both follow the change in their β. Substituting equilibrium variables into H_0 and H_1, we get

$$\frac{\tilde{W}_1}{W_1} : \frac{\tilde{W}_0}{W_0} \approx \left(\frac{u(q_{11})}{u(q_{10})} \frac{\mathfrak{S}_1}{\mathfrak{S}_1 + 1} \right)^{\tilde{\beta}_1 - \beta_1},$$

where the first and the second fractions correspond to the shifts in $H_1^{\beta_1}$ and $H_0^{\beta_0}$ respectively. If $\mathcal{L}$ is large, then, according to Proposition 1, the individual demands q_{11} and q_{10} are close to zero, and the ratio $u(q_{11})/u(q_{10})$ of the utilities in the last formula is close to 1. In our numerical example, $\mathfrak{S}_1$ is found to lie in a neighborhood of 2, and the fraction $\mathfrak{S}_1/(\mathfrak{S}_1 + 1)$ is much less than 1. These arguments are extended to economies that are characterized by moderate values of $\mathfrak{S}_1$ and small individual demands. The latter is a consequence of large values of $\mathcal{L}$ (see the discussion after Proposition 1).

The welfare of unemployed agents, as the welfare of the other individuals, undergoes an explicit positive effect from the re-distribution of spending. However, the number of unemployed agents increases (Table 1) and their income goes down (Equation (50)). This leads to a negative implicit effect. The negative effect is smaller, in general, so that

$$\frac{\tilde{W}_{n+1}}{W_{n+1}} > 1.$$

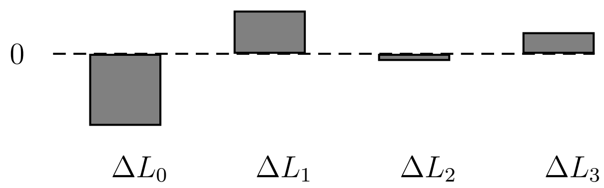

Figure 1: The difference $\Delta L_i = \tilde{L}_i - L_i$, $i = 0, 1, 2, 3$, between new and old numbers of individuals associated with sector i; the horizontal dashed line indicates 0.

Our simulation supports this observation.

The model cannot predict the changes in the employment of each specific individual. Simplifying interpretation, we assume that expanding sectors only hire workers, and vice versa. Then sector 1 attracts new workers primary from the homogeneous sector and, in exceptional cases, from the other hi-tech sector, Figure 1. A part of them is employed, whereas the rest are rejected. These rejected job market candidates become worse off, where as all the other individuals gain from the shock in demand.

Figure 1 serves only as an illustration of differences $\tilde{L}_i - L_i$, $i = 0, 1, 2, 3$, where $\tilde{L}_i$ corresponds to $\tilde{\beta}_i$. Relative changes in these differences are small for a range of parameters. In our simulation, we use $\alpha \in [0.01, 0.15]$, $\mathcal{L} \in [100, 10000]$, $C_i \in [1, 1000]$, $C_i^{\varphi} \in [0.1, 10]$, $A \in [2, 4]$, $\beta_j \in [0.1, 0.5]$, $j = 1, 2$, with $\beta_1 + \beta_2 < 0.9$.

4 Conclusion

In this paper, we describe the role of the elasticity of demand in the response of hi-tech sectors to demand shocks. The modeling is based on the Zhelobodko et al. [2012] description of the firms' decision making under monopolistic competition and additive unspecified preferences of consumers.

This description is extended to the case of heterogeneous consumers. In contrast to various studies, we do not limited ourselves to the description of symmetrical equilibriums but prove the existence of a unique equilibrium in production, if consumers are not diversified too much. The latter condition seems to be essential such that an excessive diversity can ruin the equilibrium. This unique equilibrium is predictably symmetrical, since intra-sector characteristics are identical.

In our model, firms select an optimal amount of workers among job market candidates. Job market frictions do not allow rejected agents to find another job immediately. They are limited to get an unemployment benefit. Choosing a job market, workers distinguish between sectors until the expected wages are equalized. Accepted candidates agree their wages with firms through the bargaining mechanism described in [Stole and Zwiebel, 1996]. This bargaining allows to construct a general equilibrium with the number of workers in each sector balanced endogenously (unlike numerous general equilibrium models with monopolistic competition and *preset* number of workers qualified for each available job). Such setting allows to make a theoretical prediction regarding wage inequalities and the unemployment rate. In particular, we argue that the capacity of expanding sectors is insufficient to fully absorb the inflow of workers attracted by the sector expansion and increasing demand for labor.

As far as we know, researchers debating necessity to depart from CES to VES preferences to describe certain phenomena of economic development do not estimate the size of these phenomena in their *theoretical* models. This gap is filled in this study; we argue that the influence of demand shocks on the wage differential deviates from zero in the framework of the approach and figure out the deviation.

Further development of the research could involve workers with different productivities and consumers with various utilities. Our construction of the general equilibrium is likely to survive under those extensions. A new model could contribute to problems of sorting and matching.

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A Proof of Main Results

A.1 Proof of Proposition 1

At the first stage, the existence and uniqueness of the solution of Equation (20) is established. Since $\mathfrak{S}$ depends on $Q(\xi_i)$, this Equation defines $Q(\xi_i)$ only implicitly. It is worth noting that firms tune their prices, and the demand is formed as a response to these prises. Therefore, merely the price $p(\xi_i)$ is a single but "hidden" unknown in Equation (20). Dropping product name ξ_i in the notation, we put

$$h(p) = Q(p) - C(\mathfrak{S}(p) - 1).$$

Lemma 2 states that the function $h(p)$ monotonously decreases. Then the equation $h(p) = 0$ has a unique solution. Indeed, monotonically decreasing function $h(p)$ is positive, when p is close to 0, negative, when p is large, and therefore crosses zero at some unique p. Since the functions $h(p)$ are identical within sector i, the unique $p = p_i$ depends on i but not on specification ξ_i of the product. The aggregate demand $Q_i = Q(\xi_i)$ found with this p_i equalizes the both hand sides in Equation (6). According to Lemma 3, it is well defined and symmetrical with respect to ξ_i. The other variables are also symmetrical with respect to ξ_i (Lemma 3). Therefore notation (19) neglecting the product specification is justified. Equations (20)–(29) are proved in Lemmata 3–7.

Lemma 8 verifies the second order conditions for optimization problem (1). Namely, it proves that $\partial^2\pi_i/\partial p_i^2 < 0$ at the point p_i with $\partial\pi_i/\partial p_i = 0$. Then the profit π_i attains its maximum at p_i.

Here we place five Lemmata that were used in the proof of Proposition 1.

Lemma 2. *Let Assumption 2 in the form*

$$|\sigma_i'(\varkappa)| < \frac{L_j}{2nC_i} \quad \textit{for all } i = 1, \dots, n,\ j = 0, \dots, n, \textit{ and } \varkappa \geq 0. \tag{34}$$

be valid. Then the function $h(p)$ monotonously decreases and has a unique root.

Proof. The product specification ξ_i is dropped in the formulation of the Lemma and in the Proof. Evaluating $\partial h/\partial p$ with Equations (7) and using the formula

$$\frac{\partial \mathfrak{S}}{\partial p} = -\frac{1}{pQ^2} \sum_{j=0}^{n} \sum_{k=j+1}^{n} q_j q_k (\sigma_j - \sigma_k)^2 L_j L_k - \frac{1}{pQ} \sum_{j=0}^{n} \sigma_j' \sigma_j q_j^2 L_j, \tag{35}$$

proved in lemma 20, one can see that the condition $\partial h/\partial p < 0$ is equivalent to

$$\frac{1}{p} \sum_{j=0}^{n} q_j \sigma(q_j) \left(1 - \frac{C\sigma'(q_j) q_j}{Q}\right) L_j > \frac{C}{pQ^2} \sum_{j=0}^{n} \sum_{k=j+1}^{n} q_j q_k (\sigma(q_j) - \sigma(q_k))^2 L_j L_k. \tag{36}$$

It is enough to prove

$$q_j \sigma(q_j) \left(1 - \frac{C\sigma'(q_j) q_j}{Q}\right) L_j > \frac{C}{Q^2} \sum_{k=j}^{n} q_j q_k (\sigma(q_j) - \sigma(q_k))^2 L_j L_k \quad \text{for all } j = 0, \dots, n.$$

or

$$\sigma(q_j) \left(1 - \frac{C\sigma'(q_j) q_j}{Q}\right) > \frac{C}{Q^2} \sum_{k=j}^{n} q_k (\sigma(q_j) - \sigma(q_k))^2 L_k \quad \text{for all } j = 0, \dots, n. \tag{37}$$

Among all the terms at the right hand side of (37) which are nonnegative let us choose the one delivering maximum. Let it be $q_k(\sigma(q_j) - \sigma(q_k))^2 L_k$ for some k. So, it is enough to prove that

$$\sigma(q_j) \left(1 - \frac{C\sigma'(q_j) q_j}{Q}\right) > \frac{Cn}{Q^2} q_k L_k (\sigma(q_j) - \sigma(q_k))^2 \quad \text{for all } j = 0, \dots, n. \tag{38}$$

We consider the cases of decreasing and increasing functions $\sigma(\cdot)$ separately. Without loss of generality we can assume that the individual demands $q_j, j = 1, \dots, n$ are sorted in descending or ascending order. Otherwise, they can be properly reordered. Let us first assume that $\sigma(\cdot)$ decreases and $q_j < q_k$ for $k > j$. Then $\sigma(q_j) > \sigma(q_k)$. As for $\sigma'(\cdot) < 0$ the expression

$$-\frac{C\sigma'(q_j) q_j}{Q}$$

is positive, Inequality (38) is valid, if for some $\varkappa \in [q_k, q_j]$ the following inequality is valid:

$$\sigma(q_j) > \frac{Cnq_k L_k}{Q^2} (-\sigma'(\varkappa))(q_k - q_j)\big(\sigma(q_j) - \sigma(q_k)\big). \tag{39}$$

This inequality follows from

$$\sigma(q_j) > \frac{Cnq_k L_k}{Q^2} (-\sigma'(\varkappa)) q_k \sigma(q_j),$$

or

$$1 > \frac{Cn\,(q_jL_j)\,(q_kL_k)}{L_jQ^2}(-\sigma'(\varkappa)).$$

Using the evident inequality $q_jL_j/Q < 1$, $j = 1, \ldots, n$, we claim that the last condition is valid if

$$1 > \frac{Cn}{L_j}(-\sigma'(\varkappa)),$$

following from (34).

Let now $\sigma'(\cdot) > 0$. Then, by (34),

$$0 < \sigma'(q_j)\frac{Cq_j}{Q} < \frac{L_j}{2C}\cdot\frac{Cq_j}{Q} < \frac{q_jL_j}{2Q} < \frac{1}{2},$$

The expressions in the brackets in the left hand side of Inequality (38) are between 1/2 and 1. Then Inequality (38) is valid, if

$$\sigma(q_j) > \frac{2Cn}{Q^2}q_kL_k(\sigma(q_j) - \sigma(q_k))^2 \quad \text{for all } j = 0, \ldots, n. \tag{40}$$

Deriving Inequality (40) we apply the same arguments as in the proof of Inequality (39). Indeed, the inequality $q_j > q_k$ is valid up to re-ordering of q_j. Then $\sigma(q_j) > \sigma(q_k)$. Inequality (40) is derived, if for some $\varkappa \in [q_k, q_j]$ the following inequality

$$\sigma(q_j) > \frac{2Cnq_kL_k}{Q^2}\sigma'(\varkappa)(q_j - q_k)\big(\sigma(q_j) - \sigma(q_k)\big),$$

is valid. It is weaker than inequality

$$\sigma(q_j) > \frac{Cnq_kL_k}{Q^2}(-\sigma'(\varkappa))q_j\sigma(q_j),$$

which is, in turn, weaker than

$$1 > \frac{Cn\,(q_jL_j)\,(q_kL_k)}{L_jQ^2}(-\sigma'(\varkappa)).$$

The last inequality follows from (34).

Finally, the equation $h(p) = 0$ has a unique solution. Indeed, monotonically decreasing function $h(p)$ is positive, when p is close to 0, negative, when p is large, and therefore crosses zero at some unique p. □

Lemma 3. *If the first order condition of profit maximization problem* (1) *determines a unique optimal price p_i in sector i, then this price p_i is given by Equation* (21). *The price p_i and individual demands q_{ij} for goods in sector i of workers employed in sector j are independent of the product specification ξ_i; therefore the product specification is dropped. Additionally, Equation* (20) *is valid, and the aggregate demand $Q_i = Q(\xi_i)$ is also independent of ξ_i.*

Proof. We apply a standard method to prove Lemma 3. From the first order condition ($\partial\pi(\xi_i)/\partial p(\xi_i) = 0$) we immediately obtain Equation (21). Combining it with the free entry condition ($\pi = 0$) we find that the optimal supply satisfies Equation (20). The latter equation is equivalent to the equation $h(p(\xi_i)) = 0$. According to Lemma 2, the function h has a unique root. Therefore $p(\xi_i) = p_i$ depends on i but not on ξ_i. Equations (21) and (20) relate $Q(\xi_i)$ to p_i. Then the equilibrium aggregate demand (equalled to the firms' supply) also does not depend on the product specification ξ_i, $Q(\xi_i) = Q_i$. From the first order conditions $u'(q_j(\xi_i)) = \lambda_i p_i$ of consumers' optimization problem and the monotonicity of u' it follows that the individual demands $q_j(\xi_i)$ are also symmetrical $q_j(\xi_i) = q_j(\xi_i') = q_{ij}$. $\square$

Lemma 4. *Let the first order conditions of firms' optimization problem be satisfied. Then Equations* (22)–(24), (29) *and the following relationships are valid in the equilibrium:*

$$L_i = (1-\alpha)\mathcal{L}\beta_i \frac{y_0}{y_i}, \quad i = 0, \dots, n \tag{41}$$

$$L_{n+1} = \mathcal{L}\left(1 - (1-\alpha)\sum_{j=0}^{n} \frac{\beta_j y_0}{y_j}\right), \tag{42}$$

$$L_i^u = \mathcal{L}\beta_i \left(1 - \frac{y_0}{y_i}\right) \frac{1 - (1-\alpha)\sum_{j=0}^{n} \frac{\beta_j y_0}{y_j}}{\left(1 - \sum_{j=0}^{n} \frac{\beta_j y_0}{y_j}\right)} \tag{43}$$

Proof. The revenue of firms is transmitted to their workers:

$$p_i Q_i = w_i l_i = w_i \frac{L_i}{N_i}, \quad i = 1, \dots, n. \tag{44}$$

The income of workers is

$$y_i = (1-\alpha) w_i, \quad i = 0, \dots, n. \tag{45}$$

The number of all unemployed agents is

$$L_{n+1} = \sum_{i=1}^{n} L_i^u.$$

The income of the unemployed agents is

$$y_{n+1} = \frac{\alpha}{L_{n+1}} \sum_{i=0}^{n} w_i L_i = \frac{\alpha}{(1-\alpha)L_{n+1}} \sum_{i=0}^{n} y_i L_i. \tag{46}$$

The balance of the incomes is re-written in the form:

$$L_i y_i + L_i^u y_{n+1} = L_i y_0 + L_i^u y_0. \tag{47}$$

Summing the balance of the incomes (47) up from 1 to n, we get

$$\sum_{i=1}^{n} L_i y_i + L_{n+1} y_{n+1} = y_0 \sum_{i=1}^{n} L_i + L_{n+1} y_0 = y_0(\mathcal{L} - L_0).$$

Simplifying,

$$\sum_{i=0}^{n+1} L_i y_i = \mathcal{L} y_0. \tag{48}$$

Multiplying individual budgets (4) by L_j and summing them up from $j = 0$ to $n + 1$, we have

$$p_i Q_i N_i = \beta_i \sum_{j=0}^{n+1} y_j L_j. \tag{49}$$

With Equations (44) and (48), the last Equation is transformed into (41) for $i = 1, \ldots, n$.

From (46) it follows that

$$(1-\alpha) y_{n+1} L_{n+1} = \alpha \sum_{i=0}^{n} y_i L_i$$

or

$$y_{n+1} L_{n+1} = \alpha \sum_{i=0}^{n+1} y_i L_i.$$

Applying (48) to this equation, we get

$$y_{n+1} L_{n+1} = \alpha \mathcal{L} y_0. \tag{50}$$

The following step uncovers the value of L_0. Combining Equations (46) and (41), we have

$$(1-\alpha)\alpha \mathcal{L} y_0 = \alpha \left(y_0 L_0 + \sum_{i=1}^{n} (1-\alpha)\beta_i \mathcal{L} y_0 \right).$$

Using $\beta_0 = 1 - \sum_{i=1}^{n} \beta_i$, we end up with

$$y_0 L_0 = (1-\alpha) \left(\mathcal{L} y_0 - \sum_{i=1}^{n} \beta_i \mathcal{L} y_0 \right) = (1-\alpha) \mathcal{L} y_0 \beta_0.$$

This equation is equivalent to (41), in which $i = 0$ is substituted.

The total number of unemployed agents complements the number of the all employed workers to $\mathcal{L}$:

$$L_{n+1} = \mathcal{L} - \sum_{i=0}^{n} L_i.$$

Substitution of Equations (41) into this equation leads to Equation (42).

We use Equation (47) to find L_i^u:

$$L_i^u = \frac{(y_i - y_0) L_i}{y_0 - y_{n+1}}.$$

Substituting Equations (44) and (42) into the last equation, we get

$$L_i^u = \frac{(y_i - y_0)(1-\alpha)\beta_i \mathcal{L} y_0}{y_0 \left(1 - \frac{\alpha}{1-(1-\alpha) y_0 \sum_{j=0}^{n} (\beta_j / y_j)}\right) y_i}$$

This equation is transformed into (43).

Establishing (22)–(24), we solve budget constraint (4) with respect to q_{ij}:

$$q_{ij} = \frac{\beta_i y_j}{p_i N_i}.$$

The combination of this equation with (44) and (45) results in

$$q_{ij} = \frac{\beta_i y_j Q_i}{w_i L_i} = \frac{(1-\alpha)\beta_i y_j Q_i}{y_i L_i}. \tag{51}$$

We substitute expression (51) into the definition of the aggregate demand:

$$Q_i = \sum_{j=0}^{n+1} q_{ij} L_j = (1-\alpha)\frac{\beta_i Q_i}{y_i L_i} \sum_{j=0}^{n+1} y_j L_j.$$

With Equation (48) it turns out to

$$Q_i = (1-\alpha)\frac{\beta_i Q_i}{y_i L_i} \mathcal{L} y_0.$$

Returning to Equation (51), we obtain Equations (22)-(24). Equation (29) follows from Equations (49), (48), and (45). □

Lemma 5. *Let the output Q relate to the number l of workers by the production function $Q = \theta(l - l_*)$, where $l_* > 0$ indicates a minimal labor requirement to run a firm. We assume that the firm sells its production at the price p per unit and get the profit $\pi = pQ - lw$, where w is the wage of each worker. Then the bargain between the firm and its workers, described in [Stole and Zwiebel, 1996], leads to the wages*

$$w = \theta p \frac{l^2 - l_*^2}{2l^2} + \frac{l^2 - l_*^2}{2l^2} w_0, \tag{52}$$

if the both sides of the bargain estimate alternative wages in the labor market in w_0.

Proof. Let $F(l) = \theta p(l - l_*)$ if $l > l_*$ and $F(l) = 0$ if $l \le l_*$ be the revenue of the firm as a function of labor. Then hiring $l_* + 1$ workers instead of l_*, the firm obtains the revenue $\Delta F(l_* + 1)$. Each of l_* workers benefits from this deal since their wages increase from 0 to $w(l_* + 1)$. Since the alternative of workers is to be paid w_0, their surplus is $w(l_* + 1) - w_0$. The equal devision of the surplus between the firm and each of its workers leads to the equation

$$\Delta F(l_* + 1) - (l_* + 1)w(l_* + 1) = w(l_* + 1) - w_0.$$

With a hire of the $l_* + 2$d worker, the equal division arguments result in

$$\Delta F(l_* + 2) - w(l_* + 2) + (l_* + 1)(w(l_* + 1) - w(l_* + 2)) = w(l_* + 2) - w_0.$$

Solving this Equation with respect to $w(l_* + 2)$, we get

$$w(l_* + 2) = \frac{\Delta F(l_* + 2) + (l + 1)w(l_* + 1) + w_0}{l_* + 3}.$$

In general,

$$w(l) = \frac{\Delta F(l) + (l-1)w(l-1) + w_0}{l_* + 1},$$

given $l > l_*$. The differential equation

$$w'(l) = -\frac{2}{l} + \frac{\Delta F(l) + w_0}{l}$$

corresponds to the obtained difference equation. The solution of the differential equation is

$$w = \frac{1}{l^2}\int_{l_*}^{l} (\theta p + w_0)x\,dx = \theta p \frac{l^2 - l_*^2}{2l^2} + \frac{l^2 - l_*^2}{2l^2} w_0.$$

□

Lemma 6. *Let the first order conditions of firms' optimization problem be satisfied. Then the wages of workers employed in the ith hi-tech sector are given by Equation* (25)*.*

Proof. The index i indicating the sector is dropped. Substituting Equation (21) into Equation (52) we get

$$\left(1 - \frac{\mathfrak{S}}{\mathfrak{S}-1}\frac{l^2 - l_*^2}{2l^2}\right) w = \frac{l^2 - l_*^2}{2l^2} w_0. \tag{53}$$

The zero profit condition leads to

$$pQ = lw.$$

With Equation (21) it turns to

$$\frac{\mathfrak{S}}{\mathfrak{S}-1}(l - c^{\varphi}) = l.$$

Then the optimal number of workers is

$$l = c^{\varphi}\mathfrak{S}.$$

Combining this Equation with (53), we obtain

$$\left(1 - \frac{\mathfrak{S}}{\mathfrak{S}-1}\frac{l^2 - (c^{\varphi})^2}{2l^2}\right) w = \frac{l^2 - (c^{\varphi})^2}{2l^2} w_0.$$

This equation is re-written as

$$\left(1 - \frac{\mathfrak{S}+1}{2\mathfrak{S}}\right) w = \frac{\mathfrak{S}^2 - 1}{2\mathfrak{S}^2} w_0$$

Eventually,

$$w = \frac{\mathfrak{S}+1}{\mathfrak{S}} w_0.$$

□

Lemma 7. *Let the first order conditions of firms' optimization problem be satisfied. Then the number and income of unemployed workers are given by Equations* (27) *and*

$$y_{n+1} = \frac{\alpha y_0}{\alpha + (1-\alpha)\sum_{j=1}^{n} \beta_j/(\mathfrak{S}_j + 1)} \tag{54}$$

respectively. If $\alpha \le 1$ then the income of unemployed workers is less than or equal to the income of workers employed in the homogeneous sector: $y_{n+1} \le y_0$. The equality is attained if $\alpha = 1$.

Proof. Equation (27) follows from Equation (42), Lemma 4 and Equation (25), Lemma 6. Then, substituting Equation (27), into Equation (50), we obtain Equation (54). The inequality $y_{n+1} \leq y_0$ is trivial. □

Lemma 8. *Let Assumption 1 and Condition* (17) *be valid. Then* $\partial^2\pi/\partial p^2 < 0$ *at the point* p *defined by the first order condition.*

Proof. Computing the derivative of profit (1) and taking into account the derivative $\partial Q/\partial p$ given by (7), we get

$$\frac{\partial \pi}{\partial p} = Q\left(1 - \left(1 - \frac{c^v}{p}\right)\mathfrak{S}\right).$$

Equalizing this derivative to zero, we obtain

$$\frac{p - c_v}{p}\mathfrak{S} = 1 \tag{55}$$

Computing the second derivative, we get

$$\frac{\partial^2 \pi}{\partial p^2} = \frac{\partial Q}{\partial p}\left(1 - \left(1 - \frac{c^v}{p}\right)\sigma\right) + Q\left(-\frac{c^v}{p^2} - \left(1 - \frac{c^v}{p}\right)\frac{\partial \mathfrak{S}}{\partial p}\right).$$

Substituting the equilibrium point $p = p_{\mathrm{opt}}$, i.e. using that the first derivative is zero, we obtain

$$\left.\frac{\partial^2 \pi}{\partial p^2}\right|_{p=p_{\mathrm{opt}}} = -\frac{\mathfrak{S} - 1}{p}Q - \frac{Q}{\mathfrak{S}}\frac{\partial \mathfrak{S}}{\partial p}$$

From lemma 20, equation (118) the previous formula leads to:

$$\left.\frac{\partial^2 \pi}{\partial p^2}\right|_{p=p_{\mathrm{opt}}} p = (-2\mathfrak{S} + 1)Q + \frac{1}{\mathfrak{S}}\sum_{j=0}^{n} q_j\sigma(q_j)(\sigma(q_j) + \sigma'(q_j)q_j)L_j.$$

We plan to show that the obtained expression is negative:

$$\left.\frac{\partial^2 \pi}{\partial p^2}\right|_{p=p_{\mathrm{opt}}} < 0.$$

It is equivalent to

$$\mathfrak{S}Q \cdot Q + QL\sum_{j=0}^{n} q_j\sigma(q_j)(\sigma(q_j) + \sigma'(q_j)q_j)L_j - 2\mathfrak{S}^2 Q^2 < 0. \tag{56}$$

Writing $\mathfrak{S}$ and Q as series we get

$$\sum_{j=0}^{n}\sum_{j'=0}^{n} q_j q_{j'}\sigma(q_j)(\sigma(q_j) + \sigma'(q_j)q_j + 1)L_j L_{j'} - 2\sum_{j=0}^{n}\sum_{j'=0}^{n} q_j q_{j'}\sigma(q_j)\sigma(q_j)L_j L_{j'} < 0.$$

By $T_{jj'}$ we denote the sum of *all* terms corresponding to the indices j and j':

$$T_{jj'} = q_j q_{j'}\big(\sigma^2(q_j) + \sigma^2(q_{j'}) + \sigma(q_j)(\sigma'(q_j)q_j + 1) + \sigma(q_{j'})(\sigma'(q_{j'})q_{j'} + 1) - 4\sigma(q_j)\sigma(q_{j'})\big)\gamma_j\gamma_{j'}. \tag{57}$$

From Assumption 1 it follows that $\sigma(q_j)/\sigma(q_{j'}) < 2$.

We first consider the case $\sigma' < 0$. We apply Lemma 21 with $\delta = 0$ (B is changed to 2, according to the Comment to Lemma 21) and obtain that

$$T_{jj'} < 0. \tag{58}$$

Let $\sigma' > 0$. Now we apply Lemma 21 with δ given by (18) and obtain

$$\sigma(q_j)^2 - 4\sigma(q_j)\sigma(q_{j'}) + \sigma^2(q_{j'}) + (1+\delta)\sigma(q_j) + (1+\delta)\sigma(q_{j'}) < 0.$$

Estimating $\sigma'(q_j)q_j$ and $\sigma'(q_{j'})q_{j'}$ by Inequality (18) we conclude that the brackets in (57) and $T_{jj'}$ are negative. □

Now we derive that a single firm becomes worse off, if it hires any unemployed agent (qualified for the job) and pays her even slightly more than the unemployment benefit. Let l' be an additional labor paid at w', and ΔQ be the surplus of the output (index i indicating the sector is dropped to simplify the notation). Then a considered firm has to decrease its price to some $p - \Delta p$ to shift the aggregate demand up to the level of $Q + \Delta Q$. The structure of the firm's expenses changes. The firm pays w to its first $c^v Q + c^\varphi$ workers and $w' < w$ to new l' workers that produce Q'. The profit becomes

$$\pi' = (p - \Delta p)(Q + \Delta Q) - c^v Q w - c^v Q' w' - c^\varphi w. \tag{59}$$

We are going to uncover changes in the demand share for the output of a single firm. This quantity is ill-defined because the set of firms is represented by a continuous (not discrete) set. Therefore we assume that a (small) part of firms, whose joint mass is ε, hires current unemployed agents and behaves identically.

Lemma 9. *If in the equilibrium, a small mass ε of firms in a single sector hires the same additional small number of unemployed agents proposing them identical compensations, then each of these firms becomes worse off in terms of their profit π' given by* (59).

Proof. Budget constraint (4) turns to

$$pq_j(N - \varepsilon) + (p - \Delta p)(q_j + \Delta q_j) = \beta y_j, \tag{60}$$

where index j is related to consumers. Index i indicating the sector is dropped to simplify the notation. In general, Equation (60) is valid only approximately. Indeed, new workers get a larger salary (their y_j is enlarged), and the prices for the other products also vary. However, new workers accept an arbitrary salary that is larger than the unemployment benefit. Therefore, changes in the income y_j are negligible. Changes in the other prices and corresponding outputs are also negligible, since the deviation of a small mass of firms is considered. Summing Equation (60) over all consumers, we get

$$pQ(N - \varepsilon) + (p - \Delta p)(Q + \Delta Q) = \beta Y,$$

where Y is the total amount of money in the economy. Taking into account the balance $pQN = \beta Y$, we conclude that

$$(p\Delta Q + Q\Delta p - \Delta p\Delta Q)\varepsilon = 0. \tag{61}$$

Then the revenue $(p - \Delta p)(Q + \Delta Q)$ remains pQ, and the profit

$$\pi' = pQ - c^v Qw - c^v Q'w' - c^\varphi w = -c^v Q'w' < 0$$

is negative. $\square$

A.2 Proof of Proposition 2

We put

$$M_k = \frac{\mathfrak{S}_k - 1}{\mathfrak{S}_k} \max\left\{\frac{\beta_k}{\mathfrak{S}_k}, \frac{\mathfrak{S}_k}{4(1+\mathfrak{S}_k)}\right\}, \quad k = 1, \ldots, n. \tag{62}$$

and

$$\gamma_k = \max_{j=1,\ldots,n+1} \frac{|\sigma_k'(q_{kj})| C_k M_k}{\mathcal{L}}. \tag{63}$$

Lemma 10. *Let monotonous σ_k' have an identical sign for all values of $k = 1, \ldots, n$. We assume that for all $k = 1, \ldots, n$*

$$\gamma_k \leq \frac{1}{4(2Bn+1)} \quad \text{if } \sigma_k'(q_{kj}) > 0,\ j = 1, \ldots, n+1, \tag{64}$$

$$\gamma_k \leq \frac{1}{2(2Bn+1)} \quad \text{if } \sigma_k'(q_{kj}) < 0,\ j = 1, \ldots, n+1. \tag{65}$$

If $\sigma_k'(q_{kj}) > 0$, $j = 1, \ldots, n+1$, we also assume that Assumption 2 is satisfied. Then Equation

$$\frac{\partial Q_j}{\partial \beta_i} = C_j \frac{\partial \mathfrak{S}_j}{\partial \beta_i} = K_{ji} C_j \big(\sigma_j(q_{ji}) - \sigma_j(q_{j0})\big), \quad \text{where } K_{ji} \in \left(\frac{1}{3}, 2\right). \tag{66}$$

is valid.

Proof. At the first step we write out a system of linear equation which the derivatives $\partial Q_k/\partial \beta_i$, $i, k = 1, \ldots, n$, satisfy to. Definition (6) of the aggregate elasticity of demand is evaluated with Lemma 3:

$$\mathfrak{S}_k = (1-\alpha)\sum_{j=0}^{n} \beta_j \sigma_k(q_{kj}) + \alpha\sigma_k(q_{k,n+1}), \quad k = 1, \ldots, n.$$

Then the derivative of $\mathfrak{S}_i$ with respect to β_i is given by the following formula:

$$\begin{aligned}\frac{\partial \mathfrak{S}_k}{\partial \beta_i} &= (1-\alpha)\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big) + (1-\alpha)\sum_{j=1}^{n} \beta_j \sigma_k'(q_{kj}) \frac{\partial q_{kj}}{\partial \beta_i} \\ &\quad + (1-\alpha)\left(1 - \sum_{j=1}^{n} \beta_j\right) \sigma_k'(q_{k0}) \frac{\partial q_{i0}}{\partial \beta_i} + \alpha\sigma_k'(q_{k,n+1}) \frac{\partial q_{k,n+1}}{\partial \beta_i}, \quad i, k = 1, \ldots, n.\end{aligned} \tag{67}$$

Using (20), which relates $\mathfrak{S}_k$ to Q_k, we get

$$\frac{\partial Q_k}{\partial \beta_i} = C_k \Bigg((1-\alpha)\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big) + (1-\alpha)\sum_{j=1}^{n} \beta_j \sigma_k'(q_{kj}) \frac{\partial q_{kj}}{\partial \beta_i} + (1-\alpha)\beta_0 \sigma_k'(q_{k0}) \frac{\partial q_{k0}}{\partial \beta_i} + \alpha \sigma_k'(q_{k,n+1}) \frac{\partial q_{k,n+1}}{\partial \beta_i} \Bigg), \quad i,k = 1,\dots,n. \tag{68}$$

We are going to substitute the aggregate for individual demands in Equation (68). According to (23),

$$\frac{\partial q_{k0}}{\partial \beta_i} = \frac{1}{\mathcal{L}} \frac{\partial Q_k}{\partial \beta_i}. \tag{69}$$

Similarly, from (22) we obtain that

$$\frac{\partial q_{kj}}{\partial \beta_i} = \frac{1}{\mathcal{L}} \left(\frac{\mathfrak{S}_j + 1}{\mathfrak{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{\mathfrak{S}_j^2} \frac{\partial \mathfrak{S}_j}{\partial \beta_i} \right) = \frac{1}{\mathcal{L}} \left(\frac{\mathfrak{S}_j + 1}{\mathfrak{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{C_j \mathfrak{S}_j^2} \frac{\partial Q_j}{\partial \beta_i} \right). \tag{70}$$

According to (24),

$$\frac{\partial q_{k,n+1}}{\partial \beta_i} = \frac{\alpha}{L_{n+1}} \frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha Q_k}{L_{n+1}^2} \frac{\partial L_{n+1}}{\partial \beta_i}. \tag{71}$$

The derivative of L_{n+1} is computed with (27), Proposition 1

$$\frac{\partial L_{n+1}}{\partial \beta_i} = \mathcal{L}(1-\alpha) \left(\frac{1}{\mathfrak{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{(\mathfrak{S}_j + 1)^2} \frac{\partial \mathfrak{S}_j}{\partial \beta_i} \right) = \mathcal{L}(1-\alpha) \left(\frac{1}{\mathfrak{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{C_j(\mathfrak{S}_j + 1)^2} \frac{\partial Q_j}{\partial \beta_i} \right). \tag{72}$$

Combining the last equation and Equation (71) we get:

$$\frac{\partial q_{k,n+1}}{\partial \beta_i} = \frac{\alpha}{L_{n+1}} \frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha(1-\alpha) Q_k \mathcal{L}}{L_{n+1}^2} \left(\frac{1}{\mathfrak{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{C_j(\mathfrak{S}_j + 1)^2} \frac{\partial Q_j}{\partial \beta_i} \right). \tag{73}$$

Substituting Equations (69), (70) and (73) into (68), we have

$$C_k^{-1} \frac{\partial Q_k}{\partial \beta_i} = (1-\alpha)\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big) + \frac{1-\alpha}{\mathcal{L}} \sum_{j=1}^{n} \beta_j \sigma_k'(q_{kj}) \left(\frac{\mathfrak{S}_j + 1}{\mathfrak{S}_j} \frac{\partial Q_k}{\partial \beta_i} - \frac{Q_k}{C_j \mathfrak{S}_j^2} \frac{\partial Q_j}{\partial \beta_i} \right) + \frac{1-\alpha}{\mathcal{L}} \beta_0 \sigma_k'(q_{k0}) \frac{\partial Q_k}{\partial \beta_i} + \alpha \sigma_k'(q_{k,n+1}) \left(\frac{\alpha}{L_{n+1}} \frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha(1-\alpha) Q_k \mathcal{L}}{L_{n+1}^2} \left(\frac{1}{\mathfrak{S}_i + 1} - \sum_{j=1}^{n} \frac{\beta_j}{C_j(\mathfrak{S}_j + 1)^2} \frac{\partial Q_j}{\partial \beta_i} \right) \right). \tag{74}$$

Extracting the terms containing $\frac{\partial Q_k}{\partial \beta_i}$ we have

$$C_k^{-1}\frac{\partial Q_k}{\partial \beta_i} = (1-\alpha)\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big) + \frac{(1-\alpha)\beta_k}{\mathcal{L}}\sigma_k'(q_{kk})\frac{\mathfrak{S}_k^2+1}{\mathfrak{S}_k^2}\frac{\partial Q_k}{\partial \beta_i} +$$
$$+\frac{1-\alpha}{\mathcal{L}}\sum_{\substack{j=1\\ j\neq k}}^n \beta_j\sigma_k'(q_{kj})\frac{\mathfrak{S}_j+1}{\mathfrak{S}_j}\frac{\partial Q_k}{\partial \beta_i} - \frac{1-\alpha}{\mathcal{L}}\sum_{\substack{j=1\\ j\neq k}}^n \frac{Q_k}{C_j\mathfrak{S}_j^2}\frac{\partial Q_j}{\partial \beta_i} + \frac{1-\alpha}{\mathcal{L}}\beta_0\sigma_k'(q_{k0})\frac{\partial Q_k}{\partial \beta_i}+$$
$$+\frac{\alpha^2}{L_{n+1}}\sigma_k'(q_{k,n+1})\frac{\partial Q_k}{\partial \beta_i} - \frac{\alpha^2(1-\alpha)Q_k\mathcal{L}}{L_{n+1}^2(\mathfrak{S}_i+1)}\sigma_k'(q_{k,n+1})+$$
$$+\frac{\alpha^2(1-\alpha)Q_k\mathcal{L}\beta_k}{L_{n+1}^2C_k(\mathfrak{S}_k+1)^2}\sigma_k'(q_{k,n+1})\frac{\partial Q_k}{\partial \beta_i} + \frac{\alpha^2(1-\alpha)Q_k\mathcal{L}}{L_{n+1}^2}\sigma_k'(q_{k,n+1})\sum_{\substack{j=1\\ j\neq k}}^n\frac{\beta_j}{C_j(\mathfrak{S}_j+1)^2}\frac{\partial Q_j}{\partial \beta_i}. \quad (75)$$

Grouping terms in this equation, we get

$$\underbrace{\left(C_k^{-1} - \frac{1-\alpha}{\mathcal{L}}\left(\sum_{\substack{j=1\\ j\neq k}}^n \beta_j\frac{\sigma_k'(q_{kj})(\mathfrak{S}_j+1)}{\mathfrak{S}_j} + \beta_k\sigma_k'(q_{kk})\frac{(\mathfrak{S}_k+1)^2}{\mathfrak{S}_k^2} + \beta_0\sigma_k'(q_{k0})\right)\right.}$$
$$\underbrace{\left. - \frac{\alpha^2\sigma_k'(q_{k,n+1})}{L_{n+1}}\left(1+\frac{(1-\alpha)\mathcal{L}Q_k}{L_{n+1}}\frac{\beta_k}{C_k(\mathfrak{S}_k+1)^2}\right)\right)}_{a_{kk}}\frac{\partial Q_k}{\partial \beta_i}$$
$$+\underbrace{\sum_{\substack{j=1\\ j\neq k}}^n\left((1-\alpha)\beta_j\sigma_k'(q_{kj})\frac{Q_k}{\mathcal{L}C_j\mathfrak{S}_j^2} - \frac{\alpha^2(1-\alpha)\mathcal{L}\sigma_k'(q_{k,n+1})Q_k}{L_{n+1}^2}\frac{\beta_j}{C_j(1+\mathfrak{S}_j)^2}\right)}_{a_{kj}}\frac{\partial Q_j}{\partial \beta_i}$$
$$= \underbrace{(1-\alpha)\left(\sigma_k(q_{ki}) - \sigma_k(q_{k0}) - \frac{\alpha^2\mathcal{L}\sigma_k'(q_{k,n+1})Q_k}{L_{n+1}^2(1+\mathfrak{S}_i)}\right)}_{b_k}, \quad k=1,\dots,n. \quad (76)$$

Introducing a_{kk}, a_{kj}, and b_k, as shown in the last formula, we get the system of the linear equations

$$A\begin{pmatrix}\frac{\partial Q_1}{\partial \beta_i}\\ \vdots \\ \frac{\partial Q_n}{\partial \beta_i}\end{pmatrix} = b, \quad (77)$$

where $A=(a_{kj})_{k,j=1}^n$, $b=(b_1,\dots,b_n)^T$. Thus, System (76) of linear equations with respect to $\partial Q_k/\partial\beta_i$ is found.

At the next step we are going to apply Lemma 23. First, we estimate the diagonal matrix elements a_{kk}

$$a_{kk} = C_k^{-1} - (1-\alpha)\left(\sum_{\substack{j=1\\ j\neq k}}^n \beta_j\frac{\sigma_k'(q_{kj})(\mathfrak{S}_j+1)}{\mathcal{L}\mathfrak{S}_j} + \beta_k\sigma_k'(q_{kk})\frac{(\mathfrak{S}_k+1)^2}{\mathcal{L}\mathfrak{S}_k^2} + \beta_0\frac{\sigma_k'(q_{k0})}{\mathcal{L}}\right)$$
$$- \frac{\alpha^2\sigma_k'(q_{k,n+1})}{L_{n+1}}\left(1+\frac{(1-\alpha)\mathcal{L}Q_k}{L_{n+1}}\frac{\beta_k}{C_k(\mathfrak{S}_k+1)^2}\right) \quad (78)$$

in Equations (77). We note that all the terms subtracted from C_k^{-1} in a_{kk} follow the sign of σ_k'. If $\sigma_k' \le 0$ for all $k = 0, \dots, n$, then

$$a_{kk} \ge C_k^{-1}. \tag{79}$$

We claim that if $\sigma_k' > 0$, then the term C_k^{-1} constitutes the main part of a_{kk}. To be precise, we prove that $a_{kk} > (2C_k)^{-1}$. To establish this, we evaluate the subtracting terms one by one. According to Assumption 2 and (26), that is, using $|\sigma_k'(\varkappa)| < L_j/(2C_k)$ and

$$\mathcal{L} = \frac{L_j(\mathfrak{S}_j + 1)}{(1-\alpha)\beta_j \mathfrak{S}_j}, \tag{80}$$

we have

$$\frac{\beta_j \sigma_k'(q_{kj})(\mathfrak{S}_j + 1)}{\mathcal{L}\mathfrak{S}_j} < \frac{\beta_j \sigma_k'(q_{kj})(\mathfrak{S}_j + 1)(1-\alpha)\beta_j \mathfrak{S}_j}{L_j(\mathfrak{S}_j + 1)\mathfrak{S}_j} < \\ < \frac{(1-\alpha)\beta_j^2 \sigma_k'(q_{kj})}{L_j} < \frac{\beta_j}{2C_k} \quad \text{for } j = 1, \dots, n \text{ and } j \neq k. \tag{81}$$

Further, using (80) again, we obtain:

$$\beta_k \sigma_k'(q_{kk}) \frac{(\mathfrak{S}_k + 1)^2}{\mathcal{L}\mathfrak{S}_k^2} < \frac{\beta_k \sigma_k'(q_{kk})}{L_k} \left(1 + \frac{1}{\mathfrak{S}_k}\right).$$

Since $\mathfrak{S}_k > 1$ it follows that

$$1 + \frac{1}{\mathfrak{S}_k} < 2.$$

Applying assumption (31) in the form $|\sigma_k'(\varkappa)| < L_k/(4C_k)$, we get

$$\beta_k \sigma_k'(q_{kk}) \frac{(\mathfrak{S}_k + 1)^2}{\mathcal{L}\mathfrak{S}_k^2} < \frac{\beta_k \sigma_k'(q_{kk})}{L_k} \left(1 + \frac{1}{\mathfrak{S}_k}\right) < \frac{\beta_k}{2C_k}. \tag{82}$$

From Assumption 2 it follows that

$$\frac{\beta_0 \sigma_k'(q_{k0})}{\mathcal{L}} < \frac{\beta_0 \sigma_k'(q_{k0})}{L_k} < \frac{\beta_0}{2C_k}. \tag{83}$$

Combining (81), (82) and (83), we have that

$$(1-\alpha)\sum_{\substack{j=1\\ j\neq k}}^{n} \beta_j \frac{\sigma_k'(q_{kj})(\mathfrak{S}_j + 1)}{\mathcal{L}\mathfrak{S}_j} + \beta_k \sigma_k'(q_{kk}) \frac{(\mathfrak{S}_k + 1)^2}{\mathcal{L}\mathfrak{S}_k^2} + \beta_0 \frac{\sigma_k'(q_{k0})}{\mathcal{L}} < \frac{1}{2C_k}\sum_{j=0}^{n} \beta_j < \frac{1-\alpha}{2C_k}. \tag{84}$$

Now we are going to prove that

$$\frac{\alpha^2 \sigma_k'(q_{k,n+1})}{L_{n+1}} \left(1 + \frac{(1-\alpha)\mathcal{L}Q_k\beta_k}{L_{n+1}C_k(\mathfrak{S}_k + 1)^2}\right) < \frac{\alpha}{2C_k}. \tag{85}$$

Taking into account Assumption 2 and Equation (20) we see that the last inequality follows from

$$\frac{\alpha L_k}{2C_k L_{n+1}} \left(1 + \frac{(1-\alpha)\mathcal{L}(\mathfrak{S}_k - 1)\beta_k}{L_{n+1}(\mathfrak{S}_k + 1)^2}\right) < \frac{1}{2C_k}$$

or from

$$L_k\left(\alpha+\frac{(1-\alpha)\alpha\mathcal{L}\beta_k}{L_{n+1}(\mathfrak{S}_k+1)}\right)<L_{n+1}.$$

According to (27),

$$L_{n+1}>\alpha\mathcal{L}. \tag{86}$$

Substituting (27) to the right-hand side of the last unequality we get the the evident estimate

$$L_k\left(\alpha+(1-\alpha)\frac{\beta_k}{\mathfrak{S}_k+1}\right)<\mathcal{L}\left(\alpha+(1-\alpha)\sum_{j=1}^{n}\frac{\beta_j}{\mathfrak{S}_j+1}\right)$$

that proves (85). Now the element a_{kk} given by (78) is estimated with Inequalities (84) and (85):

$$a_{kk}>\frac{1}{C_k}-\frac{1-\alpha}{2C_k}-\frac{\alpha}{2C_k}=\frac{1}{2C_k}. \tag{87}$$

Second, we estimate the non-diagonal elements a_{kj} by the diagonal element a_{jj}. Namely, we will prove that

$$|a_{kj}|<\frac{1}{2}a_{jj}\gamma\frac{\varkappa_k}{\varkappa_j} \tag{88}$$

with an appropriate choice of positive γ, $\varkappa_k$, and $\varkappa_j$. We are going to use γ given by (63) and put

$$\varkappa_j=\frac{\mathfrak{S}_j-1}{M_j}. \tag{89}$$

By (20) and (63),

$$\begin{aligned}\left|(1-\alpha)\beta_j\sigma_k'(q_{kj})\frac{Q_k}{C_j\mathcal{L}\mathfrak{S}_j^2}\right|&=(1-\alpha)\frac{|\sigma_k'(q_{kj})|C_k}{\mathcal{L}}\frac{(\mathfrak{S}_k-1)}{(\mathfrak{S}_j-1)}\frac{\beta_j(\mathfrak{S}_j-1)}{\mathfrak{S}_j^2}\frac{1}{C_j}\\&<(1-\alpha)\frac{|\sigma_k'(q_{kj})|C_kM_k}{\mathcal{L}}\frac{(\mathfrak{S}_k-1)M_j}{(\mathfrak{S}_j-1)M_k}\frac{1}{C_j}=(1-\alpha)\gamma\frac{\varkappa_k}{\varkappa_j}\frac{1}{C_j}. \end{aligned}\tag{90}$$

Dealing with the second term in a_{kj}, we estimate

$$\left(\frac{L_{n+1}}{\mathcal{L}}\right)^2\geq\left(\alpha+(1-\alpha)\frac{\beta_j}{\mathfrak{S}_j+1}\right)^2\geq4\alpha(1-\alpha)\frac{\beta_j}{\mathfrak{S}_j+1}$$

with Equation (27) and the elementary inequality $(a+b)^2\geq 4ab$. Given $\varkappa_k$ ($k=1,\ldots,n$, see (89)), γ, and the last inequality, the second term in a_{kj} is estimated with Equation (20) in the following way:

$$\begin{aligned}&\left|\frac{\alpha^2(1-\alpha)\mathcal{L}\sigma_k'(q_{k,n+1})Q_k}{L_{n+1}^2}\frac{\beta_j}{C_j(1+\mathfrak{S}_j)^2}\right|\leq\frac{|\sigma_k'(q_{k,n+1})|}{\mathcal{L}}\frac{\alpha^2(1-\alpha)Q_k}{4\alpha(1-\alpha)\beta_j}\frac{\beta_j}{C_j(1+\mathfrak{S}_j)}\\&=\alpha\frac{|\sigma_k'(q_{k,n+1})|C_k}{\mathcal{L}}\frac{(\mathfrak{S}_k-1)}{(\mathfrak{S}_j-1)}\frac{(\mathfrak{S}_j-1)}{\mathfrak{S}_j}\frac{\mathfrak{S}_j}{4(1+\mathfrak{S}_j)}\frac{1}{C_j}<\alpha\frac{|\sigma_k'(q_{k,n+1})|C_kM_k}{\mathcal{L}}\frac{(\mathfrak{S}_k-1)M_j}{(\mathfrak{S}_j-1)M_k}\frac{1}{C_j}\\&<\alpha\gamma\frac{\varkappa_k}{\varkappa_j}\frac{1}{C_j}. \end{aligned}\tag{91}$$

Now Inequality (88) is a consequence of Inequalities (90), (91), (79), and (87).

We turn to the right hand side of Equation (76). By Lemma 22,

$$\left|\sigma_k(q_{ki}) - \sigma_k(q_{k0}) - \frac{\alpha^2\mathcal{L}\sigma_k'(q_{k,n+1})Q_k}{L_{n+1}^2(1+\mathfrak{S}_i)}\right| > \frac{1}{2}\big|(\sigma_k(q_{ki}) - \sigma_k(q_{k0}))\big|.$$

Equations (22)–(23) imply that for some $\zeta, \zeta' \in (0,1)$

$$\frac{\sigma_k(q_{ki}) - \sigma_k(q_{k0})}{\sigma_{k'}(q_{k'i}) - \sigma_{k'}(q_{k'0})} = \frac{\sigma_k'(\zeta q_{ki} + (1-\zeta)q_{k0})\left(\frac{(\mathfrak{S}_i+1)Q_k}{\mathcal{L}\mathfrak{S}_i} - \frac{Q_k}{\mathcal{L}}\right)}{\sigma_{k'}'(\zeta' q_{k'i} + (1-\zeta')q_{k'0})\left(\frac{(\mathfrak{S}_i+1)Q_k'}{\mathcal{L}\mathfrak{S}_i} - \frac{Q_{k'}}{\mathcal{L}}\right)}$$
$$= \frac{\sigma_k'(\zeta q_{ki} + (1-\zeta)q_{k0})Q_k}{\sigma_{k'}'(\zeta' q_{k'i} + (1-\zeta')q_{k'0})Q_{k'}}.$$

Then

$$\left|\frac{b_k}{b_{k'}}\right| < \left|\frac{2\sigma_k'(\zeta q_{ki} + (1-\zeta)q_{k0})Q_k}{\sigma_{k'}'(\zeta' q_{k'i} + (1-\zeta')q_{k'0})Q_{k'}}.\right|$$

If $\sigma_k' > 0$ for all k, we continue in the following way:

$$\frac{b_k a_{k'j}}{b_{k'}a_{jj}} < \frac{2\sigma_k'(\zeta q_{ki} + (1-\zeta)q_{k0})Q_k}{\sigma_{k'}'(\zeta' q_{k'i} + (1-\zeta')q_{k'0})Q_{k'}}\frac{\sigma_{k'}'(q_{k'j})Q_{k'}}{L_j\mathfrak{S}_jQ_ja_{jj}} \tag{92}$$

By the definition of the constant B,

$$\left|\frac{\sigma_{k'}'(q_{k'j})}{\sigma_{k'}'(\zeta' q_{k'i} + (1-\zeta')q_{k'0})}\right| < B.$$

Using definitions (63) and (89) of γ and $\varkappa$, we claim that

$$\left|\frac{2\sigma_k'(\zeta q_{ki} + (1-\zeta)q_{k0})}{L_j}\right| < \gamma\frac{c_k^v}{2c_k^\varphi}\frac{\varkappa_k}{\varkappa_j}$$

By (87) and the last two inequalities, Inequality (92) is transformed into

$$\frac{b_k a_{k'j}}{b_{k'}a_{jj}} < \frac{2B\gamma}{\mathfrak{S}_j}\frac{\varkappa_k}{\varkappa_j} < 2B\gamma\frac{\varkappa_k}{\varkappa_j}. \tag{93}$$

Now let $\sigma_k' < 0$ for all $k = 1\ldots, n$. Then we use γ defined in Equation (63) and also end up with Inequality (93). As above, the estimates involve (65) but not (64).

Applying Lemma 23 with

$$\frac{1}{2}|\sigma_k(q_{ki}) - \sigma_k(q_{k0})| < |b_{kk}| < |\sigma_k(q_{ki}) - \sigma_k(q_{k0})|$$

and

$$\frac{1}{2C_k} < a_{kk} < \frac{3}{2C_k},$$

we conclude that the derivatives $\partial Q_k/\partial\beta_i$ ($k = 1,\ldots,n$, i is fixed) follow the sign of the differences $\sigma_k(q_{ki}) - \sigma_k(q_{k0})$ and satisfy Equation (66). $\square$

Lemma 11.
$$\frac{\partial N_i}{\partial \beta_i} = \frac{(1-\alpha)\mathcal{L}}{c_i^{\varphi}(\mathfrak{S}_i+1)}\left(1 - \frac{\beta_i K_{ii}}{\mathfrak{S}_i+1}\big(\sigma_i(q_{ii}) - \sigma_i(q_{i0})\big)\right),$$
and for $k \neq i$
$$\frac{\partial N_k}{\partial \beta_i} = -\frac{(1-\alpha)\mathcal{L}\beta_k}{c_i^{\varphi}(\mathfrak{S}_i+1)^2} K_{ki}\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big).$$
In particular, $\frac{\partial N_i}{\partial \beta_i} > 0$, and the sign of $\frac{\partial N_k}{\partial \beta_i}$ for $k \neq i$ is opposite to that of σ_k'.

Proof. From (29) it follows that
$$\frac{\partial N_i}{\partial \beta_i} = \frac{(1-\alpha)\mathcal{L}}{c_i^{\varphi}(\mathfrak{S}_i+1)}\left(1 + \frac{\beta_i}{\mathfrak{S}_i+1}\frac{\partial \mathfrak{S}_i}{\partial \beta_i}\right),$$
and for $k \neq i$
$$\frac{\partial N_k}{\partial \beta_i} = \frac{(1-\alpha)\mathcal{L}\beta_k}{c_i^{\varphi}(\mathfrak{S}_i+1)^2}\frac{\partial \mathfrak{S}_k}{\partial \beta_i}.$$
The proof follows from (66) of Lemma 10 and Assumption 2. □

Next, from (29) it follows that
$$\frac{N_k}{\beta_k \mathcal{L}} = \frac{(1-\alpha)}{c_i^{\varphi}(\mathfrak{S}_i+1)},$$
and we get

Lemma 12.
$$\frac{\partial}{\partial \beta_i}\left(\frac{N_k}{\beta_k \mathcal{L}}\right) = -\frac{(1-\alpha)}{c_k^{\varphi}(\mathfrak{S}_k+1)^2} K_{ki}\big(\sigma_k(q_{ki}) - \sigma_k(q_{k0})\big).$$
So, $\frac{N_k}{\beta_k \mathcal{L}}$ decreases if σ_k is increasing function and $\frac{N_k}{\beta_k \mathcal{L}}$ increases if σ_k is decreasing function.

To describe the change of sector output $Q_k N_k$ let us first notice that
$$Q_k N_k = \frac{(1-\alpha)\mathcal{L}\beta_k}{c_k^v}\frac{\mathfrak{S}_k - 1}{\mathfrak{S}_k+1} = \frac{(1-\alpha)\mathcal{L}\beta_k}{c_k^v}\left(1 - \frac{2}{\mathfrak{S}_k+1}\right)$$

Lemma 13. *The change of sector output is described by the following formulae*
$$\frac{\partial (Q_k N_k)}{\partial \beta_i} = \frac{2(1-\alpha)\mathcal{L}\beta_k}{c_k^v(\mathfrak{S}_k+1)^2}\frac{\partial \mathfrak{S}_k}{\partial \beta_i}$$
for $k \neq i$ and
$$\frac{\partial (Q_i N_i)}{\partial \beta_i} = \frac{Q_i N_i}{\beta_i}\left(1 + \frac{\beta_i C_i K_{ii}}{\mathcal{L}\mathfrak{S}_i(\mathfrak{S}_i+1)}\sigma_i'(\varkappa)\right)$$
Proof. The first formula of the lemma is evident. To prove the second one we can find
$$\frac{\partial (Q_i N_i)}{\partial \beta_i} = \frac{(1-\alpha)\mathcal{L}}{c_i^v}\frac{\mathfrak{S}_i - 1}{\mathfrak{S}_i+1} + \frac{2(1-\alpha)\mathcal{L}\beta_i}{c_i^v(\mathfrak{S}_i+1)^2}\frac{\partial \mathfrak{S}_i}{\partial \beta_i}$$
As for some $\varkappa \in (q_{i0}, q_{ii})$
$$\frac{\partial \mathfrak{S}_i}{\partial \beta_i} = K_{ii}\big(\sigma_i(q_{ii}) - \sigma_i(q_{i0})\big) = K_{ii}\sigma_i'(\varkappa)\big(q_{ii} - q_{i0}\big) = K_{ii}\sigma_i'(\varkappa)\frac{Q_i}{\mathcal{L}\mathfrak{S}_i} = C_i K_{ii}\sigma_i'(\varkappa)\frac{\mathfrak{S}_i - 1}{\mathcal{L}\mathfrak{S}_i}$$
$$\frac{\partial (Q_i N_i)}{\partial \beta_i} = \frac{(1-\alpha)\mathcal{L}(\mathfrak{S}_i - 1)}{c_i^v(\mathfrak{S}_i+1)}\left(1 + \frac{\beta_i C_i K_{ii}}{\mathcal{L}\mathfrak{S}_i(\mathfrak{S}_i+1)}\sigma_i'(\varkappa)\right).$$
□

Let us rewrite Equation (26) for L_k in the form

$$L_k = (1-\alpha)\beta_k \mathcal{L}\big(1 - \frac{1}{\mathfrak{S}_i + 1}\big).$$

Lemma 14. *For $k \neq i$*

$$\frac{\partial L_k}{\partial \beta_i} = \frac{(1-\alpha)\mathcal{L}\beta_k}{(\mathfrak{S}_k+1)^2}\frac{\partial \mathfrak{S}_k}{\partial \beta_i} \tag{94}$$

and

$$\frac{\partial L_i}{\partial \beta_i} = \frac{L_i}{\beta_i}\left(1 + \frac{\beta_i K_{ii} C_i(\mathfrak{S}_i - 1)}{\mathcal{L}(\mathfrak{S}_i+1)\mathfrak{S}_i^2}\sigma_i'(\varkappa)\right), \tag{95}$$

for some $\varkappa \in (q_{i0}, q_{ii})$. So, $\frac{\partial L_k}{\partial \beta_i}$ follows the sign of $\frac{\partial Q_k}{\partial \beta_i}$ and $\frac{\partial L_i}{\partial \beta_i} > 0$

Proof. The formula (94) is straitforward while the formula (95) can be obtained by the same reasoning as in lemma 13. Moreover, according to assumption 2

$$\frac{\beta_i K_{ii} C_i(\mathfrak{S}_i - 1)}{\mathcal{L}(\mathfrak{S}_i+1)\mathfrak{S}_i^2}|\sigma_i'(\varkappa)| < \frac{K_{ii}C_i}{\mathcal{L}}\frac{L_i}{2C_i} < 1.$$

The Lemma is proved. □

According to (21) and (25)

$$p_i = c_i^v w_0 \frac{\mathfrak{S}_i + 1}{\mathfrak{S}_i - 1} = p_i = c_i^v w_0 \left(1 + \frac{2}{\mathfrak{S}_i - 1}\right)$$

Lemma 15.

$$\frac{\partial p_k}{\partial \beta_i} = -\frac{2c_k^v w_0}{(\mathfrak{S}_k - 1)^2}\frac{\partial \mathfrak{S}_k}{\partial \beta_i}$$

From (72) we can get the following

Lemma 16. *If σ_i is a decreasing function then*

$$\frac{\partial L_{n+1}}{\partial \beta_i} > 0. \tag{96}$$

If σ_i is an increasing function, we assume additionally that the quantity $\mathfrak{S}^*$ defined in (32) satisfies the following condition

$$\mathfrak{S}^* < \frac{16}{1-\beta_0} - 1.$$

Then the inequality (96) is valid.

Proof. From (72) if follows that

$$\frac{\partial L_{n+1}}{\partial \beta_i} = \mathcal{L}(1-\alpha)\left(\frac{1}{\mathfrak{S}_i+1} - \sum_{j=1}^{n}\frac{\beta_j}{(\mathfrak{S}_j+1)^2}\frac{\partial \mathfrak{S}_j}{\partial \beta_i}\right).$$

So, the derivative $\frac{\partial L_{n+1}}{\partial \beta_i} > 0$, if

$$\sum_{j=1}^{n}\frac{\beta_j}{(\mathfrak{S}_j+1)^2}\frac{\partial \mathfrak{S}_j}{\partial \beta_i} < \frac{1}{\mathfrak{S}_i+1},$$

or

$$\sum_{j=1}^{n} \frac{\beta_j}{(\mathfrak{S}_j+1)^2} K_{ji}\left(\sigma_j(q_{ji}) - \sigma_j(q_{j0})\right) < \frac{1}{\mathfrak{S}_i+1},$$

$$\sum_{j=1}^{n} \frac{\beta_j}{(\mathfrak{S}_j+1)^2} K_{ji}\sigma_j'(\varkappa)\frac{C_j(\mathfrak{S}_j-1)}{\mathcal{L}\mathfrak{S}_j} < \frac{1}{\mathfrak{S}_i+1}.$$

According to Assumption 2, it is enough to prove that

$$\sum_{j=1}^{n} \frac{\beta_j}{(\mathfrak{S}_j+1)^2} K_{ji}\frac{L_i}{2C_j}\frac{C_j(\mathfrak{S}_j-1)}{\mathcal{L}\mathfrak{S}_j} < \frac{1}{\mathfrak{S}_i+1},$$

or,

$$\sum_{j=1}^{n} \frac{\beta_j(\mathfrak{S}_j-1)}{(\mathfrak{S}_j+1)^2\mathfrak{S}_j} < \frac{1}{\mathfrak{S}_i+1}. \tag{97}$$

One can check that

$$\frac{\mathfrak{S}_j-1}{(\mathfrak{S}_j+1)^2\mathfrak{S}_j} < 1/16,$$

for $\mathfrak{S}_j > 1$ so that (97) is valid if

$$\frac{1}{16}\sum_{j=1}^{n}\beta_j < \frac{1}{\mathfrak{S}_i+1},$$

or,

$$\frac{1-\beta_0}{16} < \frac{1}{\mathfrak{S}_i+1}$$

and, finally,

$$\mathfrak{S}_i < \frac{16}{1-\beta_0} - 1,$$

which proves the Lemma. □

A.3 Proof of lemma 1

Lemma 17.

$$\left(\frac{w_j}{w_0}-1\right)^{-1}\frac{\partial}{\partial\beta_i}\left(\frac{w_j}{w_0}-1\right) = -C_j\frac{K_{ji}\sigma_j'(\varkappa)(\mathfrak{S}_j-1)}{\mathcal{L}\mathfrak{S}_i\mathfrak{S}_j} \tag{98}$$

where $K_{ji}\in(1/3,2)$ is given in Lemma 10.

Proof. According to (25),

$$\frac{\partial}{\partial\beta_i}\left(\frac{w_j}{w_0}\right) = \frac{\partial}{\partial\beta_i}\left(1+\frac{1}{\mathfrak{S}_j}\right) = -\frac{1}{\mathfrak{S}_j^2}\frac{\partial\mathfrak{S}_j}{\partial\beta_i}.$$

Using Lemma 10, we get

$$\frac{\partial}{\partial\beta_i}\left(\frac{w_j}{w_0}\right) = -\frac{K_{ji}}{\mathfrak{S}_j^2}\big(\sigma_j(q_{ji}) - \sigma_j(q_{j0})\big), \quad i=1,\ldots,n.$$

With the Lagrange difference formula, we continue:

$$\frac{\partial}{\partial \beta_i}\left(\frac{w_j}{w_0}\right) = -\frac{K_{ji}}{\mathfrak{S}_j^2}\sigma_j'(\varkappa)(q_{ji}-q_{j0}),$$

for some $\varkappa \in (q_{ji}, q_{j0})$. Applying Equations (22)-(23) for the individual demands, we get:

$$\frac{\partial}{\partial \beta_i}\left(\frac{w_j}{w_0}-1\right) = -\frac{K_{ji}}{\mathfrak{S}_i\mathfrak{S}_j}\frac{\sigma_j'(\varkappa)Q_j}{\mathcal{L}}\frac{1}{\mathfrak{S}_j} = -C_j\frac{K_{ji}\sigma_j'(\varkappa)(\mathfrak{S}_j-1)}{\mathcal{L}\mathfrak{S}_i\mathfrak{S}_j}\left(\frac{w_j}{w_0}-1\right).$$

We get Equation (98). ☐

A.4 Justification of examples

Lemma 18. *Let*

$$u(x) = \int\left(\frac{\sqrt{2x+1}}{x}\right)^{1/A}dx, \quad A>1 \tag{99}$$

and

$$\mathcal{L} > \max\left\{\frac{(A+1)\big(A(2-\bar{\delta})-1\big)C_i\bar{\delta}}{A(1-\bar{\delta})}, \frac{4(2n+\bar{\delta}^2)AC_iM_i}{\bar{\delta}^2}\right\} \quad \text{for any } i,j=1,\dots,n. \tag{100}$$

We also assume that Assumption 1 is satisfied. Then Propositions 1 and 2 are valid, and

$$q_{ij} < \frac{1}{n}.$$

Comment. If $A>1$ then the utility (99) can be expressed in term of hyperelliptic functions. To be precise,

$$u(x) = Ax^{1-\frac{1}{A}}(2x+1)^{1+\frac{1}{2A}}\frac{F_1(1,2-\frac{1}{2A},2-\frac{1}{A},-2x)}{A-1},$$

where $F_1(a,b,c,z)$ is a standard notation for hyperelliptic functions (see Whittaker and Watson [1990]) for details.

Proof. A direct computation justifies that

$$\sigma(x) = -\frac{u'(x)}{u''(x)x} = A\left(2-\frac{1}{x+1}\right).$$

The main equation is

$$Q_i = C_i(\mathfrak{S}_i - 1)$$

We evaluate the right hand side for the utility in question:

$$\mathfrak{S}_i = \sum_{j=0}^{n+1}\frac{q_{ij}L_j}{Q_i}\sigma_i(q_{ij}) = A\sum_{j=0}^{n+1}\frac{q_{ij}L_j}{Q_i}\left(2-\frac{1}{1+q_{ij}}\right) = A\sum_{j=0}^{n+1}\frac{L_j}{Q_i}\left(2q_{ij}-1+\frac{1}{1+q_{ij}}\right).$$

We evaluate the obtained sum taking into account that $1/(1+\varepsilon)\sim 1-\varepsilon$. More precisely, given arbitrary $\bar{\delta}<1$, if

$$q_{ij} < \frac{1-\bar{\delta}}{\bar{\delta}},$$

then

$$1 - q_{ij} < \frac{1}{1+q_{ij}} < 1 - \bar{\delta} q_{ij}. \tag{101}$$

Returning to $\mathfrak{S}_i$, we get

$$A \sum_{j=0}^{n+1} \frac{L_j q_{ij}}{Q_i} < \mathfrak{S}_i < A \sum_{j=0}^{n+1} \frac{L_j q_{ij}(2-\bar{\delta})}{Q_i}.$$

Using $Q_i = \sum_{j=0}^{n+1} L_j q_{ij}$, we continue

$$A < \mathfrak{S}_i < A(2-\bar{\delta}).$$

In particular

$$\frac{\mathfrak{S}_i + 1}{\mathfrak{S}_i} < \frac{A+1}{A}$$

as a decreasing function with respect to $\mathfrak{S}_i$. According to the main equation $Q_i = C_i(\mathfrak{S}_i - 1)$, the output Q_i satisfies the following double inequality

$$C_i(A-1) < Q_i < C_i\big(A(2-\bar{\delta}) - 1\big).$$

Since

$$q_{ij} = \frac{Q_i(\mathfrak{S}_j + 1)}{\mathcal{L}\mathfrak{S}_j} \quad i, j = 1 \ldots, n$$

and the right hand side of this Equation increases with Q_i, it follows that

$$q_{ij} < \frac{C_i(A+1)\big(A(2-\bar{\delta}) - 1\big)}{A\mathcal{L}}. \tag{102}$$

Then inequalities

$$\mathcal{L} > \frac{(A+1)\big(A(2-\bar{\delta}) - 1\big)C_i\bar{\delta}}{A(1-\bar{\delta})} \quad \forall i, j = 1, \ldots, n \tag{103}$$

implies that $q_{ij} < (1-\bar{\delta})/\bar{\delta}$. It justifies double Inequality (101). Now we are going to check inequalities (64) and (65). They both follow from inequality

$$\max_{i=1,\ldots,n+1} \max_{\varkappa \in [\min\limits_{0\le j\le n} q_{ij}, \max\limits_{0\le j\le n+1} q_{ij}]} \frac{4|\sigma_i'(\varkappa)|C_i M_i}{\mathcal{L}} \max_{\varkappa' \varkappa'' \in [\min\limits_{0\le j\le n} q_{kj}, \max\limits_{0\le j\le n} q_{kj}]} \left(2n\frac{\sigma'(\varkappa')}{\sigma'(\varkappa'')} + 1\right) < 1. \tag{104}$$

We have already proved that $q_{ij} < (1-\bar{\delta})/\bar{\delta}$ for $i, j = 1, \ldots, n$. Then

$$\sigma'(\varkappa) = \frac{A}{(\varkappa+1)^2} > A\bar{\delta}^2, \quad \sigma'(\varkappa) < A$$

and

$$\frac{\sigma'(\varkappa')}{\sigma'(\varkappa'')} < \frac{1}{\bar{\delta}^2}.$$

Therefore Inequality (104) is weaker than

$$\max_{i=1,\ldots,n} \frac{4AC_i M_i}{\mathcal{L}} \left(\frac{2n}{\bar{\delta}^2} + 1\right) < 1.$$

It is can be re-written as

$$\mathcal{L} > \frac{4(2n+\bar{\delta}^2)AC_iM_i}{\bar{\delta}^2} \quad i = 1\ldots,n$$

We choose $\bar{\delta} = 1/(n+1)$. When $\bar{\delta}$ is changed to its largest value value $1/(1+1) = 1/2$, the last inequality becomes stronger and turns to:

$$\mathcal{L} > 32(n+1)AC_iM_i \quad i = 1\ldots,n \tag{105}$$

With $\bar{\delta} = 1/(n+1)$ Inequality (103) is weaker than

$$\mathcal{L} > 2AC_iM_iC_i(n+1). \tag{106}$$

Inequalities (105) and (106) follow from (100). Thus, Inequalities (64) and (64) are justified.

Finally, Assumption 1 is also valid. Let $\varkappa_m$ be the minimal equilibrium individual demand:

$$\varkappa_m = \min_{j=0,\ldots,n+1;i=1,\ldots,n} q_{ij}.$$

It is positive. Put,

$$\delta = \frac{1}{2}\left(\frac{\varkappa_m}{(\varkappa+1)^2} + \frac{\varkappa_m}{\varkappa+1}\right).$$

This δ agrees with Assumption 1. □

Figure 2 illustrates the behavior of three utilities (with A equalled to 1.5, 2, and 3).

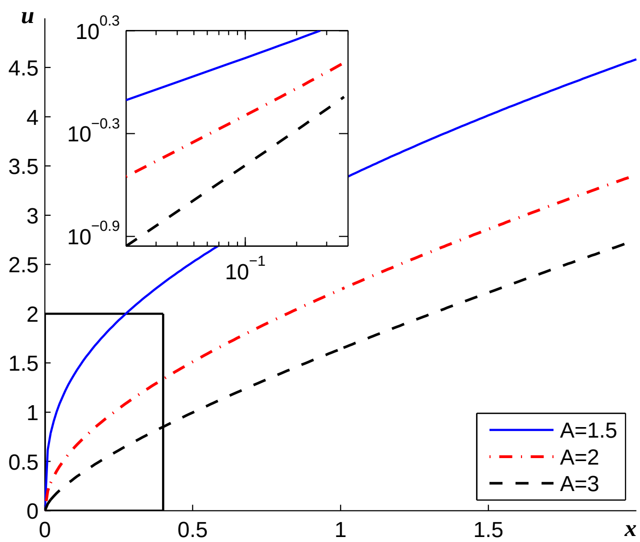

Figure 2: The utilities given by (10) with $A = 1.5$, 2, and 3; the inset: zoomed black box that corresponds to small values of the argument given in the double-logarithmic scale.

Let us now consider another family of utility examples satisfying main Assumptions

Lemma 19. *Let*

$$u(x) = \int \left(\frac{1}{x(x+2)}\right)^{1/A} dx, \quad A > 2 \tag{107}$$

and

$$\mathcal{L} > \max\left\{\frac{(2A-1)C_i(A(1+\delta)+1)\delta}{A(1-\delta^2)}, 4AC_iM_i\left(\frac{2n}{\delta^2}+1\right)\quad\right\} \quad \forall i,j = 1,\ldots,n.$$

We also assume that Assumption 1 is satisfied. Then Propositions 1 and 2 are valid, and

$$q_{ij} < \frac{1-\delta}{\delta}.$$

Comment. If $A = 2$, then

$$u(x) = \ln\left(x + 1 + \sqrt{x^2+2x}\right) \tag{108}$$

If $A > 2$ then the utility (107) can be written with hyperelliptic functions. Namely,

$$u(x) = A(x(x+2))^{\frac{A-1}{A}} \frac{F_1(1, 2-\frac{2}{A}, 2-\frac{1}{A}, -\frac{x}{2})}{2(A-1)}.$$

Proof. By direct calculation we can find

$$u'(x) = (x(x+2))^{-1/2A}, \; u''(x) = -\frac{1}{A}(x(x+2))^{-1/2A-1}(x+1)$$

$$\sigma(x) = -\frac{u'(x)}{u''(x)x} = A\left(1+\frac{1}{x+1}\right). \tag{109}$$

We are going to check now that the condition (31) is satisfied. From (109) we can find the following expression for the aggregate elastisity

$$\begin{aligned}\mathfrak{S}_i &= \sum_{j=0}^{n+1} \frac{q_{ij}L_j}{Q_i}\sigma_i(q_{ij}) = A\sum_{j=0}^{n+1}\frac{q_{ij}L_j}{Q_i}\left(1+\frac{1}{1+q_{ij}}\right) = A + \frac{A}{Q_i}\sum_{j=0}^{n+1}\frac{q_{ij}L_j}{1+q_{ij}} = \\ &= A + \frac{A}{Q_i}\sum_{j=0}^{n+1}\frac{(1+q_{ij})-1}{1+q_{ij}}L_j = A + \frac{A}{Q_i}\left(\mathcal{L} - \sum_{j=0}^{n+1}\frac{L_j}{1+q_{ij}}\right).\end{aligned} \tag{110}$$

Substituting to the demand equation

$$Q_i = C_i(\mathfrak{S}_i - 1)$$

we get

$$Q_i = \frac{AC_i}{Q_i}\left(\mathcal{L} - \sum_{j=0}^{n+1}\frac{L_j}{1+q_{ij}}\right) + C_i(A-1) \tag{111}$$

Let now δ is a number satisfying the condition $0 < \delta < 1$ and individual demand

$$q_{ij} < \frac{1-\delta}{\delta}. \tag{112}$$

Then it is easy to check that

$$1 - q_{ij} < \frac{1}{1+q_{ij}} < 1 - \delta q_{ij}. \tag{113}$$

Returning to $\mathfrak{S}_i$, we get step by step

$$\sum_{j=0}^{n+1} L_j(1-q_{ij}) < \sum_{j=0}^{n+1} \frac{L_j}{1+q_{ij}} < \sum_{j=0}^{n+1} L_j(1-\delta q_{ij}),$$

$$\mathcal{L} - \sum_{j=0}^{n+1} L_j(1-q_{ij}) > \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1+q_{ij}} > \mathcal{L} - \sum_{j=0}^{n+1} L_j(1-\delta q_{ij}),$$

$$\delta \sum_{j=0}^{n+1} q_{ij} L_j < \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1-q_{ij}} < \sum_{j=0}^{n+1} q_{ij} L_j,$$

Using $Q_i = \sum_{j=0}^{n+1} L_j q_{ij}$, we continue

$$\delta Q_i < \mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1-q_{ij}} < Q_i,$$

$$\delta A C_i < \frac{AC_i}{Q_i}\left(\mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1-q_{ij}}\right) < AC_i,$$

$$AC_i(1+\delta) - C_i < \frac{AC_i}{Q_i}\left(\mathcal{L} - \sum_{j=0}^{n+1} \frac{L_j}{1-q_{ij}} + C_i(A-1)\right) < 2AC_i - C_i.$$

According to (111), we have

$$AC_i(1+\delta) - C_i < Q_i < 2AC_i - C_i.$$

and

$$A(1+\delta) < \mathfrak{S}_i < 2A.$$

As $\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}$ is decreasing function of $\mathfrak{S}_i$ we have the following double inequality

$$\frac{2A+1}{2A} < \frac{\mathfrak{S}_i}{\mathfrak{S}_i - 1} < \frac{A(1+\delta)+1}{A(1+\delta)}$$

Next, since

$$q_{ij} = \frac{Q_i(\mathfrak{S}_j + 1}{\mathcal{L}\mathfrak{S}_j)} \quad i,j = 1\ldots,n$$

we can use the inequalities $\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i} < \frac{A(1+\delta)+1}{A(1+\delta)}$ and $Q_i < 2AC_i - C_i$ for the following relation

$$q_{ij} < \frac{(2A-1)C_i(A(1+\delta)+1)}{A(1+\delta)\mathcal{L}} \tag{114}$$

To justify (112) we solve the inequlity

$$\frac{(2A-1)C_i(A(1+\delta)+1)}{A(1+\delta)\mathcal{L}} < \frac{1-\delta}{\delta},$$

so that

$$\mathcal{L} > \frac{(2A-1)C_i(A(1+\delta)+1)\delta}{A(1+\delta)(1-\delta)}, \ \forall i,j = 1,\ldots,n \tag{115}$$

Now we are going to check Inequality (64). It follows from inequality

$$\max_{i=1,\dots,n+1} \max_{\varkappa\in[\min\limits_{0\le j\le n} q_{ij}, \max\limits_{0\le j\le n+1} q_{ij}]} \frac{4|\sigma_i'(\varkappa)|C_iM_i}{\mathcal{L}} \max_{\varkappa'\varkappa''\in[\min\limits_{0\le j\le n} q_{kj}, \max\limits_{0\le j\le n} q_{kj}]} \left(2n\frac{\sigma'(\varkappa')}{\sigma'(\varkappa'')}+1\right) < 1. \quad (116)$$

As $q_{ij} < (1-\delta)/\delta$ for $i, j = 1, \dots, n$. Then

$$|\sigma'(\varkappa)| = \frac{A}{(\varkappa+1)^2} > A\delta^2, \quad |\sigma'(\varkappa)| < A$$

and

$$\frac{\sigma'(\varkappa')}{\sigma'(\varkappa'')} < \frac{1}{\delta^2}.$$

Using $|\sigma'(\varkappa)| < A$ we get that (116) is weaker than

$$\max_{i=1,\dots,n} \frac{4AC_iM_i}{\mathcal{L}} \left(\frac{2n}{\delta^2}+1\right) < 1 \quad (117)$$

or,

$$\mathcal{L} > 4AC_iM_i\left(\frac{2n}{\delta^2}+1\right) \quad i = 1\dots, n$$

□

B Technical Lemmata

This section contains elementary computations that were used above.

Lemma 20. *The derivative $\partial\mathfrak{S}/\partial p$ is written in the following way:*

$$\frac{\partial\mathfrak{S}}{\partial p} = \frac{1}{p}\left(\mathfrak{S}^2 - \frac{1}{Q}\sum_{j=0}^{n} q_j\sigma_j(\sigma_j+\sigma_j'q_j)L_j\right). \quad (118)$$

In more details, Equation (118) *is transformed into* (35)

Proof. By Equations (6) and (5),

$$\frac{\partial\mathfrak{S}}{\partial p} = \frac{\partial}{\partial p}\left(\overbrace{\frac{1}{Q}\sum_{j=0}^{n} q_j\sigma_jL_j}^{\mathfrak{S}Q/\mathcal{L}}\right) = -\frac{1}{Q^2}\left(-\frac{Q\mathfrak{S}}{p}\right)\mathfrak{S}Q - \frac{1}{Q}\sum_{j=0}^{n}\frac{q_j}{p}\sigma_j^2L_j - \frac{1}{Q}\sum_{j=0}^{n}\frac{q_j^2}{p}\sigma_j'\sigma_jL_j,$$

where $\sigma_j = \sigma(q_j)$, $\sigma_j' = \sigma'(q_j)$. Simplifications leads to Equation (118). Applying again definition (6) and multiplying both the numerator and denominator of the second term by $Q = \sum_j q_jL_j$, we have

$$\frac{\partial\mathfrak{S}}{\partial p} = \frac{1}{pQ^2}\sum_{j=0}^{n}\left(q_j\sigma_jLj\right)\sum_{j'=0}^{n}\left(q_{j'}\sigma_{j'}Lj'\right) - \frac{1}{pQ^2}\sum_{j=0}^{n}\left(\sigma_j^2q_jL_j\right)\sum_{j'=0}^{n}\left(q_{j'}\gamma_{j'}\right) - \frac{1}{pQ}\sum_{j=0}^{n}\sigma_j'\sigma_jq_j^2L_j$$

We combine the first two terms and group the summands with identical indices to get Equation (35) □

Lemma 21. *Let B be the larger root of the equation*

$$B^2 - \left(4 - \frac{1}{y_0}\right) B + 1 + \frac{1}{y_0} = 0, \tag{119}$$

and δ be some positive number. Then for any x and y such that

$$1 + \delta \le y_0 < y < x < By \tag{120}$$

the inequality

$$x^2 - 4xy + y^2 + (1+\delta)x + (1+\delta)y < 0 \tag{121}$$

is valid. In particular, if $\delta = 0$ and $y_0 = 1$, then inequality (121) *follows from the condition*

$$1 < y < x < 2y.$$

Comment. It is worth noting that the larger root of equation (119) depends on y_0 but not too much. B increases from 2 to $2 + \sqrt{3}$ when y_0 changes from 1 to $+\infty$.

Proof. We fix an arbitrary $y > y_0$ and find the roots $x_\pm$ of the left hand side of equation (121):

$$x_\pm = \frac{4y - (1+\delta) \pm \sqrt{12y^2 - 12y(1+\delta) + (1+\delta)^2}}{2}.$$

Direct computation gives evidence that the lesser root x_- is less that y. Then for all x such that

$$y < x < \frac{4y - 1 + \sqrt{12y^2 - 12y(1+\delta) + (1+\delta)^2}}{2} \tag{122}$$

inequality (121) is valid. We intend to derive that inequality (122) follows from (120). It is enough to find the maximal B such that the inequality

$$By \le \frac{4y - (1+\delta) + \sqrt{12y^2 - 12y(1+\delta) + (1+\delta)^2}}{2}$$

is valid for all $y > y_0$. This inequality is equivalent to

$$(2B-4)y + 1 + \delta < \sqrt{12y^2 - 12y(1+\delta) + (1+\delta)^2}$$

and

$$(B^2 - 4B + 1)y^2 + (B+1)y < 0.$$

It should be valid for all y. Then the factor $B^2 - 4B + 1$ is positive, and B should be such that the inequality is valid for the smallest available y, i.e., y_0.

$$B^2 - \left(4 - \frac{1}{y_0}\right) B + 1 + \frac{1}{y_0} < 0.$$

The larger root of the corresponding equation (written in (119)) gives the required B. If $y_0 = 1 + \delta$ and $\delta = 0$, then the larger root is equal to 2. $\square$

Lemma 22. *Let the tax α satisfy the condition*

$$\alpha < \frac{1-\beta_0}{2B\mathfrak{S}^* - \mathfrak{S}^* - \beta_0}. \tag{123}$$

Then

$$\left|\sigma_k(q_{ki}) - \sigma_k(q_{k0})\right| > 2\frac{\alpha^2 \mathcal{L}|\sigma_k'(q_{k,n+1})|Q_k}{L_{n+1}^2(1+\mathfrak{S}_i)}$$

Proof. According to Lagrange's formula, the statement of the Lemma is equivalent to

$$|\sigma_k'(\varkappa)|(q_{ki} - q_{k0}) > 2\frac{\alpha^2 \mathcal{L}|\sigma_k'(q_{k,n+1})|Q_k}{L_{n+1}^2(1+\mathfrak{S}_i)},$$

where $\varkappa \in (q_{k0}, q_{ki})$. Using expressions for q_{ki} and q_{k0} from Proposition 1 we get

$$|\sigma_k'(\varkappa)|\frac{Q_k}{\mathcal{L}\mathfrak{S}_i} > 2\frac{\alpha^2 \mathcal{L}|\sigma_k'(q_{k,n+1})|Q_k}{L_{n+1}^2(1+\mathfrak{S}_i)}.$$

Existence of the constant B from (30) let us can conclude that the last inequality follows from

$$\frac{1}{B} > 2\frac{\alpha^2\mathcal{L}^2\mathfrak{S}_i)}{L_{n+1}^2(1+\mathfrak{S}_i)},$$

or, as $\alpha\mathcal{L} < L_{n+1}$, from

$$\frac{1}{B} > 2\frac{\alpha\mathcal{L}\mathfrak{S}_i}{L_{n+1}(1+\mathfrak{S}_i)},$$

Using (27) in the form

$$\frac{L_{n+1}}{\mathcal{L}} = \alpha + (1-\alpha)\sum_{j=1}^{n}\frac{\beta_j}{\mathfrak{S}_j+1}$$

we rewrite the last condition in the form

$$\alpha < \frac{1}{2B}\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}\left(\alpha + (1-\alpha)\sum_{j=1}^{n}\frac{\beta_j}{\mathfrak{S}_j+1}\right)$$

which is equivalent to

$$\alpha < \frac{\frac{1}{2B}\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}\sum_{j=1}^{n}\frac{\beta_j}{\mathfrak{S}_j+1}}{1-\frac{1}{2B}\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}\left(1-\sum_{j=1}^{n}\frac{\beta_j}{\mathfrak{S}_j+1}\right)},$$

From the definition (32) of $\mathfrak{S}^*$ and auxiliary inequality

$$\sum_{j=1}^{n}\frac{\beta_j}{\mathfrak{S}_j+1} < \frac{1}{\mathfrak{S}^*+1}\sum_{j=1}^{n}\beta_j = \frac{1-\beta_0}{\mathfrak{S}^*+1}$$

we obtain that the last condition on α is satisfied if

$$\alpha < \frac{\frac{1}{2B}\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}\frac{1-\beta_0}{\mathfrak{S}^*+1}}{1-\frac{1}{2B}\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}\left(1-\frac{1-\beta_0}{\mathfrak{S}^*+1}\right)},$$

Considering the right hand side of the last inequality as a function of $\frac{\mathfrak{S}_i+1}{\mathfrak{S}_i}$ we can conclude that its minimum value is archived at $\mathfrak{S}_i = \mathfrak{S}^*$ so that it can be rewritten in the form

$$\alpha < \frac{\frac{1}{2B}\frac{\mathfrak{S}^*+1}{\mathfrak{S}^*}\frac{1-\beta_0}{\mathfrak{S}^*+1}}{1-\frac{1}{2B}\frac{\mathfrak{S}^*+1}{\mathfrak{S}^*}\left(1-\frac{1-\beta_0}{\mathfrak{S}^*+1}\right)} = \frac{1-\beta_0}{2B\mathfrak{S}^*\left(1-\frac{\mathfrak{S}^*+1}{2B\mathfrak{S}^*}+\frac{1-\beta_0}{2B\mathfrak{S}^*}\right)},$$

which is equivalent to (123). ∎

Lemma 23. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \quad \text{with } a_{ii} > 0,\ i = 1, \dots, n;$$

$$\tilde{A} = \begin{pmatrix} b_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ b_3 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \cdots & \tilde{a}_{2n} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \cdots & \tilde{a}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \tilde{a}_{n3} & \cdots & \tilde{a}_{nn} \end{pmatrix}.$$

The system of the linear equations

$$Ax = b$$

is considered. We assume that first, the right hand side is not dispersed:

$$|b_{j_1} a_{ij_2}| < |b_i a_{j_2 j_2}| B\gamma \frac{\varkappa_{j_1}}{\varkappa_{j_2}} \quad j_1 \neq i,\ j_2 \neq i, \tag{124}$$

and, second, the elements a_{ii}, $i = 1, \dots, n$, being positive, dominate the other elements:

$$|a_{ij}| < \gamma a_{jj} \frac{\varkappa_i}{\varkappa_j}, \quad i \neq j, \quad \gamma < \frac{1}{(2B+1)n}. \tag{125}$$

for some positive numbers $\varkappa_1$, ..., $\varkappa_n$, Then the sign of each root x_i follows the sign of b_i, $i = 1, \dots, n$, and

$$\frac{1}{2}\frac{|b_i|}{a_{ii}} < |x_i| < \frac{3}{2}\frac{|b_i|}{a_{ii}}.$$

Proof. The roots x_1, x_2, ..., x_n of the equation $Ax = b$ are determined by Cramer's rule:

$$x_1 = \frac{|\tilde{A}|}{|A|}.$$

For simplicity, we assume that $b_1 > 0$. The determinant $|\tilde{A}|$ is defined as the sum of the terms $a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n}$, where the sequences $\{i_k\}_{k=1}^n$ and $\{j_k\}_{k=1}^n$ are permutations of $\{1, \dots, n\}$. We are going to establish that the sign of the determinant follows the sign of the main diagonal term:

$$D_b = b_1 a_{22} a_{33} \dots a_{nn}.$$

Let k be an arbitrary integer between 0 and $k-2$, and T_k be the sum of the terms

$$\tilde{a}_{i_1 j_1} \dots \tilde{a}_{i_{n-k} j_{n-k}} \tilde{a}_{i_{n-k+1} i_{n-k+1}} \dots \tilde{a}_{i_n i_n}, \quad i_l \neq j_l \text{ for } l = 1, \dots, n-k \tag{126}$$

that contain merely k diagonal elements of the matrix $\tilde{A}$. This k obviously cannot be equal to $n-1$ because in this case the n-th multiplier in T_{n-1}, "choosing" its own row and column, ultimately coincides with the last main diagonal element. We reconsider the multipliers in (126) in such a way that the first index (= the row of the matrix element) of the current multiplier coincides with the second index (= the column) of the following multiplier. Given term (126), we define a map r transforming the first index of each matrix element in (126) into the second index: $r(i_l) = j_l$. Then starting from $\tilde{a}_{i_1,r(i_1)}$ we select $\tilde{a}_{r(i_1),rr(i_1)}$, $\tilde{a}_{rr(i_1),rrr(i_1)}$, and so forth. At some step ν, the sequence of indices i_1, $r(i_1)$, $rr(i_1)$, ..., forms a cycle: the last index coincides with the first index, $\underbrace{r\dots r}_{\nu \text{ times}}(i_1) = i_1$. If $\nu = n-k$, then the non-diagonal multipliers in (126) form a single cycle. If $\nu < n-k$, the non-diagonal multipliers form several cycles. We distinguish terms (126) with and without $b_{i_1} \neq b_1$. Let $b_{i_1} \neq b_1$ be in term (126). Then, by (124),

$$|b_{i_1} a_{1i_2}| < |b_1 a_{i_2 i_2}| < B\gamma \frac{\varkappa_{i_1}}{\varkappa_{i_2}},$$

where $i_2 = r(1) = rr(i_1)$. Other multipliers in (126) is estimated with (125). Eventually,

$$|\tilde{a}_{i_1 j_1} \dots \tilde{a}_{i_{n-k} j_{n-k}} \tilde{a}_{i_{n-k+1} j_{n-k+1}} \dots \tilde{a}_{i_n j_n}| < D_b B \gamma^{n-k-1}.$$

The factors $\varkappa_{i_1}$, $\varkappa_{i_2}$, ..., disappear altogether because the indices in (126) are split into cycles. Given indices i_{n-k+1}, ..., i_n, the number of different terms (126) is equal to $(n-k-1)!$ (this number coincides with the number of the terms in the determinant of the $(n-k) \times (n-k)$-matrix, when the main diagonal has been already chosen). The number of possibilities to choose indices i_{n-k+1}, ..., i_n among numbers 2, ..., n (the first multiplier in the product is chosen among b_i, $i = 1, \dots, n$) is $\binom{n-1}{k}$. Let T_{k1} be the sum of all terms (126) with b_{i_1}, $i_1 \neq 1$. Then

$$\sum_{k=0}^{n-2} |T_{k1}| < \sum_{k=0}^{n-2} D_b B \gamma^{n-k-1} (n-k-1)! \binom{n-1}{k} = D_b B \sum_{k=0}^{n-2} \gamma^{n-k-1} (n-1)(n-2)\dots(k+1). \quad (127)$$

If term (126) contains b_1 (instead of b_{i_1} with $i_1 \neq 1$), then

$$\tilde{a}_{i_1 j_1} \dots \tilde{a}_{i_{n-k} j_{n-k}} \tilde{a}_{i_{n-k+1} j_{n-k+1}} \dots \tilde{a}_{i_n j_n} < D_b \gamma^{n-k}.$$

Let T_{k2} be the sum of all terms (126) with b_1. The same arguments as above lead to

$$\sum_{k=0}^{n-2} |T_{k2}| < \sum_{k=0}^{n-2} D_b \gamma^{n-k-1} (n-k-1)! \binom{n-1}{k-1} = D_b \sum_{k=0}^{n-2} \gamma^{n-k} \frac{(n-1)(n-2)\dots k}{n-k}. \quad (128)$$

Let $c = 1/(3B)$. Then combining (127) and (128), we end up with $T_k = T_{k1} + T_{k2}$ given by the following equation:

$$\sum_{k=0}^{n-2} |T_k| < D_b \left(\frac{cB}{1-c} + \frac{c^2}{2(1-c)} \right).$$

If $B > 2$, the last inequality can be simplified into

$$\sum_{k=0}^{n-2} |T_k| < \frac{1}{2} D_b B.$$

In the same manner one can argue that the determinant $|A|$ is greater than $a_{11} a_{22} \dots a_{nn}/2$. Then the sign of x_1 coincides with the sign of b_1. The same arguments are valid for other variables x_2, ..., x_n. $\square$

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